Commutators Of θ −Types Generalized Calderón-Zygmund Operator On Generalized Weighted Morrey

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ABSTRACT. In this paper, we conduct a comprehensive study on the boundedness properties of commutators generated by a BMO function and generalized Calderón-Zygmund operator. Specifically, we analyze their mapping behavior from a generalized weighted Morrey space to another generalized weighted Morrey spaces under some conditions on the parameters. Our main objective is to establish novel and refined conditions on the pair of parameter functions that guarantee the boundedness of these commutators on the spaces. The findings presented in this work contribute to a deeper understanding of the interplay between function spaces, weight conditions, and the structure of commutators in harmonic analysis.

Keywords: commutator, generalized Calderón-Zygmund operator, generalized weighted Morrey Space, A_p weight

1. INTRODUCTION

Morrey spaces $\mathcal{M}_q^p$ for $1 \leq p < q < \infty$ is defined as the set of any locally integrable functions f such that

$$\|f\|_{\mathcal{M}_q^p} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{-q}}} \left(\frac{1}{B(a,r)} \int_{B(a,r)} |f(x)|^p dx\right)^{\frac{1}{p}}$$

is finte. The function spaces were first introduced by C. B. Morrey in 1938 [13]. Many authors then studied commutators of a certain operator on the spaces [5], [12]. Generalized Morrey spaces were then introduced as investigated in [13]. In 2009, Komori and Shirai [11]introduced the weighted Morrey space $L^{p,\kappa}(w)$. The two spaces $L^{p,\varphi}$ and $L^{p,\kappa}(w)$ were generalized by Guliyev *et al* [10] as generalized weighted Morrey space $L^{p,\varphi}(w)$.

Calderón-Zygmund operator plays important roles in harmonic analysis. The operator was introduced by R. Coifman and Y. Meyer in 1978 [2]. In 1985, K. Yabuta [7] introduced the generalized Calderón-Zygmund operator, namely Calderón-Zygmund operator of type $\omega(t)$ and of type $(\log, \omega(t))$. Many authors then studied the operators on certain spaces [21], [20].

The result by Guliyev implies the boundedness of commutators of Calderón-Zygmund operator from generalized weighted Morrey space $\mathcal{M}_{\psi_1}^{p,w}$ to generalized weighted Morrey space $\mathcal{M}_{\psi_2}^{p,w}$ provided $1 < p < \infty, w \in A_p$, and the pair (ψ_1, ψ_2) satisfying certain conditions [10]. Unfortunately, the conditions investigated in the study contain the logaritmic natural function which is technically difficult to handle. In this paper, we investigated the conditions for the pair ψ_1 and ψ_2 where the logaritmic natural function does not apper and still ensure the boundedness of generalized Calderón-Zygmund operator on generalized weighted Morrey space. Through this paper, we give new conditions for the pair ψ_1 and ψ_2 that ensure the boundedness of commutator of generalized Calderón-Zygmund operator (immediately imply for Calderón-Zygmund operator).

2. PRELIMINARIES: SOME DEFINITIONS AND PREVIOUS RESULTS

Let $a \in \mathbb{R}^n$ and $r > 0$. We denote $B(a,r)$ as an open ball centered at a with radius r. For the Ball $B = B(a,r)$ and $k > 0$, kB denotes $B(a,kr)$, namely the ball with the same center as B but with radius k times r. Moreover, $|E|$ denotes the Lebesgue measure of a measurable subset E of $\mathbb{R}^n$. A weight w is a nonnegative locally integrable functions on $\mathbb{R}^n$ taking values in the interval $(0,\infty)$ almost everywhere [11].

For the weight w, $1 \leq p < \infty$, and a measurable subset E of $\mathbb{R}^n$, we define the weighted Lebesgue spaces $L^{p,w}(E)$ over E by the set of any functions f on E such that the norm $\|f\|_{L^{p,w}(E)}$ is finite where

$$\|f\|_{L^{p,w}(E)} = \left(\int_E |f(y)|^p w(y)dy\right)^{\frac{1}{p}}. \qquad (2.1)$$

If $E = \mathbb{R}^n$, then we write $L^{p,w} = L^{p,w}(E) = L^{p,w}(\mathbb{R}^n)$.

Next, we give some definitions and well-known results used in this paper.

Definition 2.1 *(A_p weight)* Let $1 < p < \infty$, we define A_p as a set of all weights w on $\mathbb{R}^n$ for which there exists a constant $C > 0$ such that [8]

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx\right) \times \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C \qquad (2.2)$$

for all balls $B(a,r)$ in $\mathbb{R}^n$.

Theorem 2.1 For $1 < p < \infty$ and $w \in A_p$, there exists $C > 0$ such that [8], [16]

$$\frac{w(B)}{w(E)} \leq C \left(\frac{|B|}{|E|}\right)^p \qquad (2.3)$$

for all balls B and measurable sets $E \subseteq B$ where $w(B) = \int_B w(x)\,dx$.

Definition 2.3 $BMO = BMO(\mathbb{R}^n)$ is set of all locally integrable functions b such that [4], [6]

$$\|b\|_* = \sup_{B=B(a,r)} \frac{1}{|B|}\int_B |b(y) - b_B|\,dy < \infty \qquad (2.4)$$

where $b_B = \frac{1}{|B|}\int_B b(y)\,dy$.

Definition 2.4 (Generalized Calderón-Zygmund Operator) Let θ be a nonegative and increasing function on $(0,\infty)$ that satisfies [7]

$$\int_0^1 \theta(t)\frac{dt}{t} < \infty. \qquad (2.5)$$

Suppose $T = T_\theta$ be a bounded linear operator $\mathcal{S}$ to $\mathcal{S}'$ which satisfies L^2-boundedness and for all infinitely diferentiable functions f with compact support we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \in (\text{supp}(f))^C$$

where K is a function on $\mathbb{R}^n \times \mathbb{R}^n$ except for the diagonal $\{(x,x) : x \in \mathbb{R}^n\}$ such that there exists a constant $A > 0$ for which

$$|K(x,y)| \leq \frac{A}{|x-y|^n}, \qquad x \neq y$$

and

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq A\frac{1}{|x-y|^n}\theta\left(\frac{|x-z|}{|x-y|}\right),$$

for

$$|x-z| < \frac{|x-y|}{2}.$$

The operator T is called generalized Calderón-Zygmund operator of $\theta-$type or simply Calderón-Zygmund operator of $\theta-$type. If we set $\theta(t) = t^\delta$ where $\delta > 0$, then we have that $T = T_\theta$ is (classical) Calderón-Zygmund Operator (see [9], [3]) that was first introduced by Coifman and Meyer [2].

Definition 2.5 (Commutator) Let b a locally integrable function defined on $\mathbb{R}^n$. For linear operator T, we define the commutator of T by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x) \qquad (2.6)$$

for $x \in \mathbb{R}^n$.

We rewrite the following results for $[b,T]$ and $[b,I_\alpha]$ on weighted Lebesgue spaces as well as the properties of $b \in BMO$. Note that the first following theorem based on the results in [1].

Theorem 2.6 Let $b \in BMO$ and T be a Calderón-Zygmund operator of $\theta-$type. If $1 < p < \infty$ and $w \in A_p$, then $[b,T]$ is bounded on $L^{p,w}$.

Theorem 2.7 [17] Let $b \in BMO$ and T be a Calderón-Zygmund operator. If $1 < p < \infty$ and $w \in A_p$, then $[b,T]$ is bounded on $L^{p,w}$.

We see that Therem 2.6 is an extension of Theorem 2.6.

Theorem 2.8 Let $b \in BMO$. Then, there is a constant $C > 0$ such that for all ball $B = B(a,r)$ in $\mathbb{R}^n$ and $j \in \mathbb{R}^n$ [18],

$$|b_{2^{j+1}B} - b_B| \leq C \cdot (j+1)\|b\|_*. \qquad (2.7)$$

Theorem 2.9 Let $b \in BMO$ and $1 < p < \infty$. Then, there is a constant $C > 0$ such that for all $B = B(a,r)$ in $\mathbb{R}^n$ and $w \in A_p$ [19],

$$\left(\int_B |b(y) - b_B|^p w(y)dy\right)^{\frac{1}{p}} \leq C\,\|b\|_* w(B)^{\frac{1}{p}} \qquad (2.8)$$

We now present the definition of generalized weighted Morrey spaces. The spaces will become the spaces of our main interest in this paper.

Definition 2.10 (Generalized Weighted Morrey Space) Let $1 \leq p < \infty, w \in A_p$, and ψ be a positive function on $\mathbb{R}^n \times (0,\infty)$. $\mathcal{M}_{\psi}^{p,w} = \mathcal{M}_{\psi}^{p,w}(\mathbb{R}^n)$ is set of all measurable functions f such that the norm $\|f\|_{\mathcal{M}_{\psi}^{p,w}}$ is finite where

$$\|f\|_{\mathcal{M}_{\psi}^{p,w}} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \left(\frac{1}{w(B(a,r))} \int_{B(a,r)} |f(x)|^p w(x)dx \right)^{\frac{1}{p}} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))}.$$

(2.10)

3. MAIN RESULTS

In this sections we state our main results regarding the boundedness of commutator generated by a BMO function and generalized Calderón-Zygmund operator on generalized weighted Morrey spaces.

Theorem 3.1 Let $1 < p < \infty, w \in A_p$, and T be Calderón-Zygmund operator of θ −type. Suppose that ψ_1 and ψ_2 are positive functions on $\mathbb{R}^n$ such that there exist $C > 0$ and $\alpha > 0$ such that

$$\sup_{(a,r)\in\mathbb{R}^n\times(0,\infty),\lambda\geq 2} \frac{\lambda^{\alpha}}{\psi_2(a,r)} \times \int_{\lambda r}^{\infty} \frac{\inf_{s\leq t<\infty} \psi_1(a,t)\, w(B(a,t))^{\frac{1}{p}}}{w(B(a,s))^{\frac{1}{p}}} \frac{ds}{s} \leq C.$$

If $b \in BMO$, then $[b,T]$ is bounded from $\mathcal{M}_{\psi_1}^{p,w}$ to $\mathcal{M}_{\psi_2}^{p,w}$. Precisely, there is a constant $D > 0$ such that

$$\|[b,T]f\|_{\mathcal{M}_{\psi_2}^{p,w}} \leq D\|b\|_* \|f\|_{\mathcal{M}_{\psi_1}^{p,w}}$$

for $f \in \mathcal{M}_{\psi_1}^{p,w}$.

Suppose that there is a constant $C > 0$ and $\beta > 0$ such that

$$\psi_1(a,r)s^{\beta} \leq C\, \psi_1(a,r)r^{\beta},$$

for $(a,r) \in \mathbb{R}^n \times (0,+\infty)$. Then, for all $\lambda \geq 2$ and $a \in \mathbb{R}^n$ we have that

$$\begin{aligned}\int_{\lambda r}^{\infty} \psi_1(a,s)\frac{ds}{s} &= \int_{\lambda r}^{\infty} \psi_1(a,s)s^{\beta}\frac{ds}{s^{\beta+1}} \\ &\leq \psi_1(a,r)r^{\beta} \int_{\lambda r}^{\infty} \frac{ds}{s^{\beta+1}} \\ &\leq \psi_2(a,r)r^{\beta}\frac{1}{\alpha}\frac{1}{\lambda^{\beta}}\frac{1}{r^{\beta}} \\ &= \frac{1}{\lambda^{\beta}}\psi_1(a,r).\end{aligned}$$

Therefore, we have the following theorems by [20] which may be viewed as the corollary of Theorem 3.1.

Theorem 3.2 [20] Let $w \in A_p$ where $1 < p < \infty$ and $T = T_{\theta}$ be a Calderón-Zygmund operator of θ −type. Suppose that there is a constant $C > 0$ and $\beta > 0$ such that

$$0 < \psi(a,s)s^{\beta} \leq C\, \psi(a,r)r^{\beta},$$

for $(a,r) \in \mathbb{R}^n \times (0,\infty)$. If $b \in BMO$, then there is $D > 0$ such that

$$\|[b,T]f\|_{\mathcal{M}_{\psi}^{p,w}} \leq D\|b\|_* \|f\|_{\mathcal{M}_{\psi}^{p,w}}$$

for $f \in \mathcal{M}_{\psi}^{p,w}$.

4. PROOF OF THE MAIN RESULTS

We prove the main results presented in Section 3 in this section. Before proving the main results, we provide some lemmas.

Lemma 4.1 Suppose that u_1, u_2, v_1, and v_2 are positive functions on $\Omega \times R_{\Omega}$. If there exists $\alpha > 0$, g: $\mathbb{R}^n \times (0,\infty)$ is increasing, and

$$\int_{\lambda r}^{\infty} \frac{\inf_{t\leq s<\infty} u_1(a,s)v_1(a,s)}{u_2(a,t)} \frac{dt}{t} \leq C\frac{1}{\lambda^{\alpha}} v_2(a,r)$$

where

$$(a,r) \in \mathbb{R}^n \times (0,\infty), \lambda > 2,$$

then

$$\int_{\lambda r}^{\infty} u_2(a,t)^{-1} g(a,t)\frac{dt}{t} \leq Cv_2(a,r)\frac{1}{\lambda^{\alpha}} \sup_{a\in\mathbb{R}^n, t>0} \frac{g(a,t)}{u_1(a,t)v_1(a,t)}$$

where

$$(a,r) \in \mathbb{R}^n \times (0,\infty).$$

Proof. By the assumption, we have that

$$\int_{r}^{\infty} u_2(a,t)^{-1} g(a,t)\frac{dt}{t}$$

$$= \int_r^\infty \frac{\inf_{t\le s<\infty} u_1(a,s)v_1(a,s)}{\inf_{t\le s<\infty} u_1(a,s)v_1(a,s)} u_2(a,t)^{-1} g(a,t)\frac{dt}{t}$$
$$\le \int_r^\infty \frac{\inf_{t\le s<\infty} u_1(a,s)v_1(a,s)}{u_2(a,t)}$$
$$\times \sup_{t<s<\infty} \frac{1}{u_1(a,s)v_1(a,s)} g(a,s)\frac{dt}{t}$$
$$\le \int_r^\infty \frac{\inf_{t\le s<\infty} u_1(a,s)v_1(a,s)}{u_2(a,t)}\frac{dt}{t}$$
$$\times \sup_{(a,t)\in\mathbb{R}^n\times(0,\infty)} \frac{g(a,t)}{u_1(a,t)v_1(a,t)}$$
$$\le C v_2(a,r)\frac{1}{\lambda^\alpha} \sup_{(a,t)\in\mathbb{R}^n\times(0,\infty)} \frac{g(a,t)}{u_1(a,t)v_1(a,t)}$$

for $(a,r)\in\mathbb{R}^n\times(0,\infty)$. This proves the Lemma ■

The following lemmas are proved in [15].

Lemma 4.2 Let $1<p<\infty, w\in A_p$. For each $(a,s)\in\mathbb{R}^n\times(0,\infty)$ we have that

$$\frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|\,dy \le \frac{C_1}{w\big(B(a,s)\big)^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,s))}$$

where C_1 is constant that is independent of a and s.

Lemma 4.3 Let φ a nonnegative function on $\mathbb{R}^n\times(0,\infty)$ such that the map $r\mapsto\varphi(a,r)$ is increasing for all $a\in\mathbb{R}^n$. Then, for each $1<p<\infty, w\in A_p$ and the ball $B(a,r)$ we have

$$\varphi(a,r)\le C\, w\big(B(a,r)\big)^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s}$$

where $C>0$ is independent of $a\in\mathbb{R}^n$ and $r>0$.

Next, we prove our main results.

Proof of Theorem 3.1

Let $a\in\mathbb{R}^n$ and $r>0$. We write $B=B(a,r)$ and

$$f=\sum_{k=0}^\infty f_k$$

where

$$f_0=f\cdot\mathcal{X}_{2B}$$

and

$$f_k=f\cdot\mathcal{X}_{2^{k+1}B\setminus 2^kB}$$

for $k\in\mathbb{N}$. By the definition of the commutator on f_k,

$$[b,T]f_k(x)= b(x)Tf_k(x)-T(bf_k)(x)$$

where $x\in\mathbb{R}^n$. Hence, we may write the following estimate for $[b,T]f_k(x)$ and $k\in\mathbb{N}$

$$|[b,T]f_k(x)| \le |b(x)-b\_B|\,|Tf_k(x)| + |T([b_B-b_{2^{k+1}B}]f_k)(x)| + |T([b_{2^{k+1}B}-b]f_k)(x)|.$$

We shall estimates the three terms on the right hand side in the last inequality. Let $x\in B$. For the first term, if $y\in(2B)^c$, then $|x-y|\sim|a-y|$ and

$$|b(x)-b_B||Tf_k(x)|$$
$$\le C|b(x)-b_B|\int_{\mathbb{R}^n}\frac{|f_k(y)|}{|x-y|^n}dy$$
$$\le C|b(x)-b_B|\int_{2^{k+1}B\setminus 2^kB}\frac{|f(y)|}{|a-y|^n}dy$$
$$\le C|b(x)-b_B|\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y)|\,dy$$
$$\le C|b(x)-b_B|\frac{1}{w(2^{k+1}B)^{\frac{1}{p}}}\|f\|_{L^{p,w}(2^{k+1}B)}.$$

Theorem 2.9 then implies that

$$\|(b-b_B)Tf_k\|_{L^{p,w}(B)}\le C\frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}}\left(\int_B|b(x)-b_B|^p w(x)dx\right)^{\frac{1}{p}}$$
$$\le C\|b\|_* w(B)^{\frac{1}{p}}\frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}}.$$

For the second inequality, by Theorem 2.8 we have

$$\left|T\big(|b_B-b_{2^{k+1}B}|f_k\big)(x)\right|$$
$$\le C\int_{\mathbb{R}^n}\frac{|f_k(x)||b_B-b_{2^{k+1}B}|}{|x-y|^n}dy$$
$$\le C\left|b_B-b_{2^{(k+1)}B}\right|\int_{\mathbb{R}^n}\frac{|f_k(y)|}{|a-y|^n}dy$$
$$\le C\|b\|_*(k+1)\frac{1}{|2^{(k+1)}B|}\int_{2^{k+1}B}|f(y)|\,dy.$$

This implies that

$$\left\|T\big(|b_B-b_{2^{k+1}B}|f_k\big)\right\|_{L^{p,w}(B)} \le C\|b\|_*(k+1)w(B)^{\frac{1}{p}}\frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}}.$$

For the last term, by A_p condition, Holder inequality, the fact that $w^{-\frac{p'}{p}} \in A_{p'}$ and by Theorem 2.9 we have that

$$\left|T\left(\left|b_B - b_{2^{k+1}B}\right| f_k\right)(x)\right| \leq C\int_{\mathbb{R}^n} \frac{\left|b_{2^{k+1}B} - b(y)\right| |f_k(y)|}{|x-y|^n} dy \leq C\int_{2^{k+1}B\setminus 2^k B} \frac{\left|b_{2^{k+1}B} - b(y)\right| |f_k(y)|}{|a-y|^n} dy$$
$$\leq C\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} \left|b_{2^{k+1}B} - b(y)\right| |f(y)| dy \leq C\frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{|2^{k+1}B|} \times \left\|\left(b_{2^{k+1}B} - b(\cdot)\right) w^{-\frac{1}{p}}\right\|_{L^{p'}(2^{k+1}B)}$$
$$\leq C\frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{|2^{k+1}B|}\|b\|_* w^{-\frac{p'}{p}}(2^{k+1}B)^{\frac{1}{p'}} = C\|b\|_* \frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}} \times w^{-\frac{p'}{p}}(2^{k+1}B)^{\frac{1}{p'}} \frac{w(2^{k+1}B)^{\frac{1}{p}}}{|2^{k+1}B|} \leq C\|b\|_* \frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}}.$$

Then,

$$\left\|T\left(\left|b_B - b_{2^{k+1}B}\right| f_k\right)\right\|_{L^{p,w}(B)} \leq C\|b\|_* w(B)^{\frac{1}{p}} \frac{\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}}.$$

Therefore, by the estimates of the three terms,

$$\|[b,T]f_k\|_{L^{p,w}(B)} \leq C\|b\|_* w(B)^{\frac{1}{p}} \frac{(k+3)\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}}.$$

Note that by Lemma 4.1,

$$\int_{\lambda r}^{\infty} \frac{\|f\|_{B(a,t)}}{w(B(a,t))^{\frac{1}{p}}}\frac{dt}{t} \leq C\frac{1}{\lambda^\alpha}\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}\psi_2(a,r)$$

for $a \in \mathbb{R}^n$, $r > 0$, and $\lambda \geq 2$. By the boundedness of $[b,T]$ on $L^{p,w}$,

$$\|[b,T]f_0\|_{L^{p,w}(B)} \leq \|[b,T]f_0\|_{L^{p,w}} \leq C\|b\|_*\|f_0\|_{L^{p,w}} = C\|b\|_*\|f\|_{L^{p,w}(2B)}.$$

By Lemma 4.3 and the last two estimate,

$$\|[b,T]f_0\|_{L^{p,w}(B)} \leq C\|b\|_* w(2B)^{\frac{1}{p}}\int_{2r}^{\infty} \frac{\|f\|_{B(a,t)}}{w(B(a,t))^{\frac{1}{p}}}\frac{dt}{t} \leq C\|b\|_* w(2B)^{\frac{1}{p}}\frac{1}{2^\alpha}\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}\psi_2(a,r) = C\|b\|_* w(2B)^{\frac{1}{p}}\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}\psi_2(a,r).$$

Hence, by Theorem 2.2 and the definition of the norm of generalized weighted Morrey spaces, we obtain that

$$\|[b,T]f_0\|_{\mathcal{M}^{p,w}_{\psi_2}} = \sup_{a\in\mathbb{R}^n, r>0}\frac{1}{\psi_2(a,r)}\frac{\|[b,T]f_0\|_{L^{p,w}(B(a,r))}}{w(B(a,r))^{\frac{1}{p}}} \leq C\sup_{a\in\mathbb{R}^n, r>0}\frac{1}{\psi_2(a,r)} \times \frac{\|b\|_* w(2B)^{\frac{1}{p}}\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}\psi_2(a,r)}{w(B(a,r))^{\frac{1}{p}}} \leq C\|b\|_*\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.$$

In the other hand, we obtain from the estimate for the $L^{p,w}(B)$ norm on $[b,T]f_k$ that

$$\|[b,T]f_k\|_{L^{p,w}(B)} \leq C\|b\|_* w(B)^{\frac{1}{p}}\frac{(k+3)\|f\|_{L^{p,w}(2^{k+1}B)}}{w(2^{k+1}B)^{\frac{1}{p}}} \leq C(k+3)\|b\|_* \times w(B)^{\frac{1}{p}}\int_{2^{(k+1)}r}^{\infty}\frac{\|f\|_{L^{p,w}(B(a,t))}}{w(B(a,t))^{\frac{1}{p}}}\frac{dt}{t} \leq C(k+3)\|b\|_*\frac{w(B)^{\frac{1}{p}}}{(2^{k+1})^\alpha}\|f\|_{M^{p,w}_{\psi_1}}\psi_2(a,r)$$

which implies from the definition of the norm of generalized weighted Morrey spaces that

$$\|[b,T]f_k\|_{\mathcal{M}^{p,w}_{\psi_2}} = \sup_{a\in\mathbb{R}^n, r>0}\frac{1}{\psi_2(a,r)}\frac{\|[b,T]f_k\|_{L^{p,w}(B(a,r))}}{w(B(a,r))^{\frac{1}{p}}} \leq C\frac{(k+3)}{(2^{k+1})^\alpha}\|b\|_*\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \times \sup_{a\in\mathbb{R}^n, r>0}\frac{1}{\psi_2(a,r)}\frac{w(B)^{\frac{1}{p}}\psi_2(a,r)}{w(B(a,r))^{\frac{1}{p}}} = C\frac{(k+3)}{(2^{k+1})^\alpha}\|b\|_*\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.$$

By the estimate for $\|[b,T]f_0\|_{\mathcal{M}^{p,w}_{\psi_2}}$ and $\|[b,T]f_k\|_{\mathcal{M}^{p,w}_{\psi_2}}$ for any $k \in \mathbb{N}$ as well as the fact that

$$\sum_{k=1}^{\infty} \frac{(k+3)}{(2^{k+1})^{\alpha}} < \infty,$$

we have that

$$\begin{aligned}
\|[b,T]f\|_{\mathcal{M}^{p,w}_{\psi_2}} &\leq C \sum_{k=0}^{\infty} \|[b,T]f_k\|_{\mathcal{M}^{p,w}_{\psi_2}} \\
&= C\left[\|[b,T]f_0\|_{\mathcal{M}^{p,w}_{\psi_2}} + \sum_{k=0}^{\infty} \|[b,T]f_k\|_{\mathcal{M}^{p,w}_{\psi_2}}\right] \\
&\leq C\left[\|b\|_* \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} + \sum_{k=1}^{\infty} \frac{(k+3)}{(2^{k+1}\, r)^{\alpha}} \|b\|_* \|f\|_{\mathcal{M}^{p,w}_{\psi_1}}\right] \\
&= C\|b\|_* \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \left[1 + \sum_{k=1}^{\infty} \frac{(k+3)}{(2^{k+1}\, r)^{\alpha}}\right] \\
&= C\|b\|_* \|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.
\end{aligned}$$

This proves that $[b,T]$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ as desired ■

5. CONCLUSION

From the results, we may conclude that the commutator $[b,T]$ is bounded from the generalized weighted Morey spaces $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ under some conditions. The conditions obtained do not contain the logaritmic natural function.

6. REFERENCES

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