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The Locating Chromatic Number of Pizza Graphs Containing a Single Path on the Outer Vertices

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ABSTRACT
In graph theory, the locating chromatic number refers to the minimum number of colours required to colour the vertices of a graph such that each vertex is uniquely identifiable by its own colour and the colours of its neighbouring vertices. This concept is associated with graph coloring, which entails assigning colors to a graph's vertices in such a way that no two adjacent vertices have the same color. It determines the minimum number of colours necessary to produce a proper vertex colouring. This research focuses on determining the locating chromatic number of Pizza graphs containing a single path on the outer vertices.

KEYWORDS
Locating chromatic number, Pizza graphs, Single path.

1. INTRODUCTION

Graph theory is a core area of discrete mathematics that examines the relationships between objects using vertices and edges. One of the important topics in graph theory is graph coloring, which leads to various extensions such as locating chromatic numbers. This paper investigates the locating chromatic number of a specific class of graphs, called pizza graphs, which have a single path on their outer vertices.

Chartrand et al. [1] initiated the investigation of the partition dimension in connected graphs as an innovative approach to addressing the challenges associated with determining the metric dimension of graphs. Their work opened new perspectives on exploring how to partition the vertex set of a graph to uniquely identify the locations of vertices based on distance vectors, thereby extending the concept of the metric dimension. The metric dimension has practical relevance in various applications, such as guiding autonomous robots in network navigation by enabling precise localization within a graph structure [2], as well as in the classification of chemical data, where it assists in generating compelling and distinct representations for sets of chemical compounds to ensure accurate differentiation among them [3]. Through these studies, the exploration of metric and partition dimensions continues to make significant contributions to the development of graph-based models in both theoretical and applied contexts.

The idea of a locating chromatic number plays a significant role in graph theory and was initially proposed by Chartrand et al. [3] in 2002. It focuses on assigning colours to graph vertices in such a way that each vertex can be uniquely identified based on the colours of its neighbours. This concept has been utilized in various fields, including robotics and chemical data analysis.

The locating chromatic number was originally introduced by Chartrand et al in 2002 [3]. It combines two key concepts in graph theory: vertex coloring and the partition dimension of a graph. The locating chromatic number of a graph is formally defined as follows: Let $U = (V, E)$ be a graph that is connected and e be a proper w-coloring of U with color $\{1, 2, \ldots, w\}$. Let $\Pi = \{K_1, K_2, \ldots, K_w\}$ be a partition of $V(U)$ which is induced by coloring e. The color code $e_\Pi(u)$ of u is the ordered w-tuple $(d(u, K_1), d(u, K_2), \ldots, d(u, K_w))$ where $d(u, K_l) = min\ \{d(u, x) : x \in K_l\}$ for any $l \in 1, 2, \ldots, w$. Suppose all distinct vertices of U have distinct colour codes. In that case, e is called w-locating colouring of U. The locating chromatic number, symbolized by $\chi_L(U)$, is the smallest w such that U has a locating w-coloring.

The following section presents various results related to the locating chromatic number of graphs [4]. This study aims to develop a systematic approach for calculating the locating chromatic number associated with origami graphs and their corresponding subdivisions, with a focus on analyzing a special category referred to as barbell origami graphs [5], the subdivided form of a particular barbell origami graph [6] and steps to find the locating chromatic number of an Origami graph with Python programming [7]. Asmiati et al. [8] initiated the exploration of the locating chromatic number under specific operations on origami graphs, investigating the locating chromatic number resulting from the corona operation between a path graph and a cycle graph [9]. Sudarsana et al.[10] determined the locating chromatic number of the m-shadow of complete multipartite graphs and paths.

Among the various types of graphs studied in this context, the pizza graph presents a visually distinctive structure that resembles slices of pizza. This study specifically focuses on a variation of the pizza graph that features a single path formed by its outer vertices, where peripheral vertices are connected in a linear sequence. Such a structure presents unique challenges in determining the locating-chromatic number, as the distribution of vertices and the distances between color classes become more complex.

To date, investigations into the locating chromatic number of pizza graphs have remained limited, particularly in cases involving special structures, such as a single path along the outer vertices. Therefore, a more in-depth study is needed to determine both the upper bounds and exact values of the locating-chromatic number for this class of graphs. The findings from this research are expected to contribute theoretically to the advancement of graph coloring concepts and to enhance our understanding of their applications in non-standard graph structures. The following pizza graph definition is taken from [11]. A pizza graph P_{Z_x} is a graph with $V\ (P_{Z_x}) = \{p, q_l, r_l : l \in \{1, \ldots, x\}\}$ and $E\left(P^1_{Z_x}\right) = pq_l, q_l r_l : l \in \{1, \ldots, x\} \cup \{r_l r_{l+1} : l \in \{1, \ldots, x-1\} \cup \{r_x r_1\}$.

2. PRELIMINARIES

In this section, we provide a concise yet essential overview of several fundamental definitions and theorems that are crucial for establishing a clear understanding of the theoretical framework underlying our research. These foundational concepts serve as the core references that will guide and support the analysis and interpretation of our primary results in the subsequent sections.

The locating chromatic number of a graph, which refines the standard chromatic number by incorporating location-based distinctions among vertices, is defined as follows:

Definition 2.1. [3] Let $U = (V,\ E)$ be a graph that is connected and e be a proper w-coloring of U with color $\{1,\ 2,\ \ldots,\ w\}$. Let $\Pi = \{K_1,\ K_2,\ \ldots,\ K_w\}$ be a partition of $V(U)$ which is induced by coloring e. The color code $e_\Pi(u)$ of u is the ordered w-tuple $(d(u,\ K_1), d(u,\ K_2),\ \ldots,\ d(u,\ K_w))$ where $d\,(u,\ K_l) = min\ \{d\,(u,\ x) :\ x \in K_l\}$ for any $l\ \in\ 1,\ 2,\ \ldots,\ w$. Suppose all distinct vertices of U have distinct colour codes. In that case, e is called w-locating colouring of U. The locating chromatic number, symbolised by $\chi_L(U)$, is the smallest w such that U has a locating w-coloring.

The subsequent definition of pizza graphs is adopted from the work of Nabila and Salman:

Definition 2.2. [11] A pizza graph P_{Z_x} is a graph with $V\ (P_{Z_x}) = \{p, q_l, r_l : l \in \{1, \ldots, x\}\}$ and $E\left(P^1_{Z_x}\right) = pq_l, q_l r_l : l \in \{1, \ldots, x\} \cup \{r_l r_{l+1} : l \in \{1, \ldots, x-1\} \cup \{r_x r_1\}$

Next, pizza graphs that have a single path on the outer vertices will be defined as follows:

Definition 2.3. A pizza graphs that have a single path on the outer vertices $P^1_{Z_x}$ is a graph with is a graph with $V\ \left(P^1_{Z_x}\right) = \{p, q_l, r_l,\ s_l : l \in \{1, \ldots, x\}\}$ and $E\left(P^1_{Z_x}\right) =\ \{pq_l, q_l r_l, r_l s_l : l \in \{1, \ldots, x\}\} \cup \{r_l r_{l+1} : l \in \{1, \ldots, x-1\}\} \cup \{r_x r_1\}$

The subsequent theorems serve as fundamental principles for establishing the lower bound of the locating chromatic number of a graph. The collection of neighboring vertices of a vertex j in U is represented by $N(j)$.

Theorem 2.1. [3] Suppose e represents a locating coloring in a connected graph B. If i and j are distinct vertices of B such that $d\,(i,\ z) = d(j, z)$ for all $z \in V(B) - i, j$, then $e(i) \neq e(j)$. In particular, if i and j are non-adjacent vertices such that $N(i) \neq N(j)$, then $e(i) \neq e(j)$.

Theorem 2.2. [12] The locating chromatic number of the pizza graph is 4 for $s = 3$ and s for $s \geq 4$.

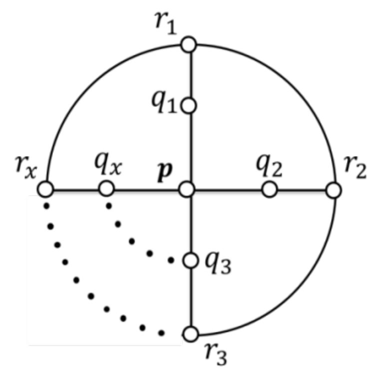

Figure 1. A Pizza Graph

3. RESULT & DISCUSSION

The following section discusses the locating chromatic number of pizza graphs containing a single path on the outer vertices

Theorem 3.1. Let $P^1_{Z_x}$ be a pizza graph that has a single path on the outer vertices for $x \geq 3$. Then:

$$\chi_L\left(P^1_{Z_x}\right) = \begin{cases} 4, & for\, x = 3 \\ x, & otherwise \end{cases}$$

Proof. A pizza graphs that have a single path on the outer vertices $P^1_{Z_x}$ is a graph with $V\ (P_{Z_x}) = \{p, q_l, r_l : l \in \{1,\ldots,x\}\}$ and $E\left(P^1_{Z_x}\right) = \{pq_l, q_lr_l, r_ls_l : l \in \{1,\ldots,x\}\} \cup \{r_lr_{l+1} : l \in \{1,\ldots,x-1\}\} \cup \{r_xr_1\}$. First, we establish a lower bound for the locating chromatic number of Pizza graphs containing a single path on the outer vertices. Given that $P^1_{Z_x}$ contains P_{Z_x}, it follows from **Theorem 2.2** that the locating chromatic number admits a lower bound of 4 when $x = 1$, and a lower bound of x when $x \geq 4$. In order to establish an upper bound for pizza graphs featuring a single path on their outer vertices, we analyze the problem by distinguishing between two specific cases.

Case 1. For $x = 3$

We now proceed to establish that the upper bound for $\chi_L\left(P^1_{Z_x}\right)$ is at most 4. To demonstrate that $\chi_L\left(P^1_{Z_x}\right) \leq 4$, we define a colouring k employing four colors in the following way :

$$\begin{aligned} K_1 &= \{q_3, r_1, s_3\}; \\ K_2 &= \{q_1, r_2, s_1\}; \\ K_3 &= \{q_2, r_3, s_2\}; \\ K_4 &= \{p\}. \end{aligned}$$

With the application of coloring t, we derive the color codes for $V(P^1_{Z_x})$ as follows:

$$\begin{aligned} t_\Pi(p) &= (1,\ 1,\ 1,\ 0); \\ t_\Pi(q_1) &= (1,\ 0,\ 2,\ 1); \\ t_\Pi(q_2) &= (2,\ 1,\ 0,\ 1); \\ t_\Pi(q_3) &= (0,\ 2,\ 1,\ 1); \\ t_\Pi(r_1) &= (0,\ 1,\ 1,\ 2); \\ t_\Pi(r_2) &= (1,\ 0,\ 1,\ 2); \\ t_\Pi(r_3) &= (1,\ 1,\ 0,\ 2); \\ t_\Pi(s_1) &= (1,\ 0,\ 2,\ 3); \\ t_\Pi(s_2) &= (2,\ 1,\ 0,\ 3); \\ t_\Pi(s_3) &= (0,\ 2,\ 1,\ 3); \end{aligned}$$

Clearly, if all the vertices in $P^1_{Z_x}$ have distinct color codes, then t is a locating coloring. So $\chi_L\left(P^1_{Z_x}\right) \leq 4$.

Case 2. For x $\geq$ 4

Subsequently, to show that x serves as an upper bound for the locating chromatic number of Pizza graphs containing a single path on the outer vertices $P^1_{Z_x}$, it is enough to establish the existence of a locating coloring k: $V\ (P^1_{Z_x}) \rightarrow 1,\ 2,\ ...,\ x$. For $x \geq 4$, we formulate the function k as follows:

$$k(p) = s$$
$$k(q_l) = \begin{cases} l+1, & \text{for } l \in \{1,\ldots,x-2\}, \\ 1, & \text{for } l \in \{x-1,\ldots,x\}. \end{cases}$$
$$k(r_l) = l, \quad \text{for } l \in \{1,\ldots,x\}$$
$$k = \begin{cases} l+1, & \text{for } l \in \{1,\ldots,x-2\}, \\ 1, & \text{for } l \in \{x-1,\ldots,x\}. \end{cases}$$

With the application of coloring k, we derive the color codes for $V(P^1_{Z_x})$ as follows:

$$k_\Pi(p) = \begin{cases} 0, & \text{for } x^{th} \text{ ordinate}, \\ 1, & \text{otherwise.} \end{cases}$$
$$k_\Pi(q_l) = \begin{cases} 0, & \text{for } (l+1)^{th} \text{ ordinate},\ l \in \{1,\ldots,x-2\}, \\ 1, & \text{for } l^{th} \text{ ordinate},\ l \in \{1,\ldots,x\}, \\ 1, & \text{for } 1^{st} \text{ ordinate},\ l \in \{x-1,\ldots,x\}, \\ 2, & \text{for } x^{th} \text{ ordinate},\ l \in \{1,\ldots,x-1\}, \\ 2, & \text{otherwise.} \end{cases}$$
$$k_\Pi(r_l) = \begin{cases} 0, & \text{for } l^{th} \text{ ordinate},\ l \in \{1,\ldots,x\}, \\ 1, & \text{for } (l+1)^{th} \text{ ordinate},\ l \in \{1,\ldots,x-1\}, \\ 1, & \text{for } (l-1)^{th} \text{ ordinate},\ l \in \{x-2,\ldots,x\}, \\ 1, & \text{for } x^{th} \text{ ordinate},\ l = 1, \\ 2, & \text{for } (l+2)^{th} \text{ ordinate},\ l \in \{1,\ldots,x-2\}, \\ 2, & \text{for } (l-2)^{th} \text{ ordinate},\ l \in \{2,\ldots,x-1\},\ x \geq 5, \\ 2, & \text{for } (x-1)^{th} \text{ ordinate},\ l = 1, \\ 2, & \text{for } x^{th} \text{ ordinate},\ l \in \{2,\ldots,x-2\}, \\ 2, & \text{for } 2^{nd} \text{ ordinate},\ l = n, \\ 3, & \text{otherwise.} \end{cases}$$
$$k_\Pi(s_l) = \begin{cases} 0, & \text{for } (l+1)^{th} \text{ ordinate},\ l \in \{1,\ldots,x-2\}, \\ 1, & \text{for } 1^{st} \text{ ordinate},\ l \in \{x-1,\ldots,x\}, \\ 1, & \text{for } l^{th} \text{ ordinate},\ l \in \{1,\ldots,x\}, \\ 1, & \text{for } x^{th} \text{ ordinate},\ l = x, \\ 3, & \text{for } x^{th} \text{ ordinate},\ l \in \{2,\ldots,x-2\}, \\ 2, & \text{otherwise.} \end{cases}$$

Since the color codes of all vertices in $V\left(P^1_{Z_x}\right)$ are distinct. Then the coloring k is a locating coloring. So $\chi_L\left(P^1_{Z_x}\right) \leq x$. Accordingly, the proof is now complete □

Figure 2 provides an example of $\chi_L\left(P^1_{Z_8}\right) = 8$.

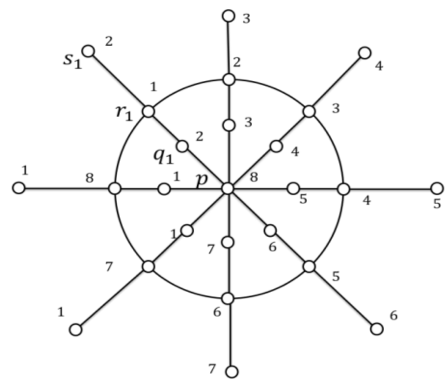

Figure 2. A minimum locating coloring of $P^1_{Z_8}$

4. CONCLUSION

This research examines the chromatic number of Pizza graphs containing a single path on the outer vertices. By constructing a colouring scheme and analysing the colour codes for each vertex, it was determined that the locating chromatic number is 4 when $x = 3$ and x for $x \geq 4$. These findings support the advancement of locating chromatic number theory within the context of pizza graphs. They can serve as a basis for further research on other variations of pizza graphs with different outer path structures or with additional imposed conditions on these paths.

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