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Convolution and Correlation Forms for the Offset Coupled Fractional Fourier Transform

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ABSTRACT
This work presents convolution and correlation in offset coupled fractional Fourier transform. The offset coupled fractional Fourier transform can be regarded as a generalized version of the coupled fractional Fourier transform. Various properties including this work like inversion formula and Parseval are studied in general and in detail for the offset coupled fractional Fourier transform. Furthermore, this work establishes the relationship between the two-dimensional Fourier transform and the offset coupled fractional Fourier transform.

KEYWORDS
Convolution and correlationoff, Fractional Fourier transform, Set coupled fractional Fourier transform.

1. INTRODUCTION

Fractional Fourier transform (FrFT), as with the traditional Fourier transform (FT), is an effective mathematical tool that has been used extensively in quantum mechanics, neural networks, differential equations, optics, pattern recognition, radar, sonar, and other communication systems; see, for examples, [1],[2],[3],[4], [5] considered an extension of the traditional FT started in 1980 by Namias[6].

This work focus on the Coupled offset fractional Fourier Transform. Under certain conditions, we first obtain the symmetry properties of the offset Coupled fractional Fourier Transform for real signals. A new form of convolution and correlation theorem associated with this transform is proposed. In this study, the definition of convolution and correlation will be developed to obtain convolution and correlation theorems for the Coupled offset fractional Fourier transform.

The motivation for this study also arises from practical needs in modern communication and imaging systems, where signals often undergo various forms of coupling and displacement. For example, radar systems operating under Doppler shift and spatial offsets naturally benefit from OCFrFT analysis. Matched filtering in radar traditionally performed via time-domain convolution or frequency-domain multiplication can be enhanced using the FrFT. Researchers have shown that using an optimal fractional order improves chirp pulse compression and signal to noise ratio, surpassing classical FT approaches[7],[8]. In sonar and underwater acoustics, similar benefits have been demonstrated through improved detection of linear frequency-modulated signals via FrFT-based correlation techniques[9].

Likewise, in optical systems featuring angular modulation, the OCFrFT framework provides a natural means to account for phase shifts and coupling effects between spatial and frequency dimensions. The rotated time-frequency representation inherent in FrFT is particularly useful for filter design and signal analysis in such non-stationary settings. Within these contexts, a properly formulated convolution theorem aligned with OCFrFT is crucial. It lays the theoretical foundation for matched filters, system identification algorithms, and adaptive signal-processing frameworks that handle offset and coupling intricacies.

As a preliminary step toward our main results, we first review the mathematical structure of the OCFrFT, highlighting the effects of coupling parameters and offset terms. After that, we explore the impact of these modifications on symmetry properties, leading to the identification of key constraints for real signal processing. Based on this foundation, we develop the new convolution and correlation definitions and prove the corresponding theorems. Our proofs are based on operator theory and integral-kernel representations, ensuring consistency with both traditional FT and fractional frameworks[10],[11].

Recent developments have significantly enriched the theoretical framework of fractional and coupled transforms. The uncertainty principles for the coupled fractional Fourier transform have been explored in various forms, emphasizing bounds and stability under signal transformations[12]. Additional properties and structural extensions of the coupled FrFT were introduced in [13], offering deeper insight into their operational behavior. Beyond the fractional domain, related mathematical tools such as the wavelet transform have provided complementary time-frequency analysis techniques [14], while time-frequency foundations are comprehensively presented in [15]. The classical Fourier transform and its widespread applications remain a cornerstone of signal analysis, as detailed in [16].

In the context of windowed transforms, important properties of the windowed linear canonical transform and its logarithmic uncertainty principles have been studied in [17], with further investigation on its connection to the windowed Fourier transform and broader uncertainty frameworks in [18]. These concepts are crucial in understanding signal concentration and resolution in both standard and fractional domains [19].

Furthermore, the logarithmic, Heisenberg, and short-time uncertainty principles associated with the fractional Fourier domain have been rigorously analyzed in [20], with applications extended to the short-time linear canonical transform of complex signals [21]. Recent work also introduced new mathematical relations within the coupled FrFT domain [22] and established uncertainty principles tailored for quadratic-phase Fourier transforms, providing a broader spectrum for theoretical generalization [23].

The remainder of this paper is organized as follows. Section 2 presents a comprehensive review of the offset coupled fractional Fourier transform, including its definition, kernel representation, and the influence of its parameters. Section 3 introduces the convolution and correlation operations adapted to the OCFrFT, together with formal statements of theorems and their corresponding proofs. Section 4 concludes the paper and discusses potential directions for future research.

We believe that the formulation of convolution and correlation theorems in the OCFrFT domain not only enriches the mathematical theory of fractional transforms but also opens up new possibilities for signal processing in complex, non-stationary environments. As systems continue to evolve toward higher dimensions and exhibit more sophisticated interactions, transform frameworks like the OCFrFT backed by solid theoretical tools will become increasingly essential.

2. PRELIMINARIES

We first review the relevant material related to the fractional Fourier transform (FrFT) and the coupled fractional Fourier transform and its basic properties, which will be required in the sequel. We start with the well-known definition.

Definition 2.1. This study proposes a convolution and correlation theorem for measurable functions on $\mathbb{R}^2$ defined Lebesgue space $L^\infty(\mathbb{R}^2) - norm$, as follows:

$$\|f\|_{L^r(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |f(z)|^r \, dz\right)^{\frac{1}{r}} < \infty, \quad 1 \leq r < \infty \tag{1}$$

Where $z = (z_1, z_2) \in \mathbb{R}^2$, $dz = dz_1 dz_2$.

Specifically, if $r = \infty$ we get the $L^\infty(\mathbb{R}^2)$–norm

$$\|f\|_{L^\infty(\mathbb{R}^2)} = \operatorname*{ess\,sup}_{z\in\mathbb{R}^2} |f(z)| \tag{2}$$

If f is continuous, then the above equation can be written as

$$\|f\|_{L^\infty(\mathbb{R}^2)} = \sup_{z\in\mathbb{R}^2} |f(z)| \tag{3}$$

In general, the inner product of $L^2(\mathbb{R}^2)$ is defined as

$$\langle f, h\rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} f(z)\, \overline{g(z)}\, dz \tag{4}$$

Definition 2.2. Given a function $f \in L^2(\mathbb{R}^2)$, the two-dimensional fractional Fourier transform with parameter α is defined by the form

$$\mathscr{F}_\alpha\{f\}(\zeta) = \int_{\mathbb{R}^2} f(z)\, K_\alpha(\zeta, z)\, dz, \tag{5}$$

where the kernel $K_\alpha(\zeta, z)$ is defined as

$$K_\alpha(\zeta, z) = \begin{cases} A_\alpha\, e^{-i\frac{\cot\alpha}{2}\left(|z|^2+|\zeta|^2\right) - iz\cdot\zeta \csc\alpha}, & \alpha \neq n\pi, \\ \delta(z-\zeta), & \alpha = 2n\pi, \\ \delta(z+\zeta), & \alpha = (2n+1)\pi, \quad n \in \mathbb{Z}. \end{cases} \tag{6}$$

For a Dirac delta function

$$\delta(z-\zeta) = \delta(z_1-\zeta_1)\,\delta(z_2-\zeta_2),$$

and

$$A_\alpha = \frac{1 - i\cot\alpha}{2\pi}, \qquad \overline{A_\alpha} = \frac{1 + i\cot\alpha}{2\pi}. \tag{7}$$

It is easy to verify that the FrFT kernel satisfies the following basic properties

$$\overline{K_\alpha(\zeta, z)} = K_{-\alpha}(\zeta, z),$$

and

$$\int_{\mathbb{R}^2} K_\alpha(z, \zeta)\, \overline{K_\alpha(z, \zeta')}\, dz = \delta(\zeta - \zeta').$$

Where $\overline{K_\alpha(\zeta, z)}$ is the conjugate of $K_\alpha(\zeta, z)$.

3. MAIN RESULT

In this section, we will discuss the construction of Coupled offset fractional Fourier Transform and then convolution and correlation will be established to get the main result of this paper which will be stated in the form of theorem.

3.1 Coupled Offset Fractional Fourier Transformation

In this section, we will discuss the construction of Coupled offset fractional Fourier Transform and then study some properties that will be used to obtain the main results of this paper.

Definition 3.1. Suppose $f \in L^2(\mathbb{R}^2)$. The coupled fractional Fourier transform is defined as:

$$M^\gamma_{\alpha,\beta}\{f\}(\zeta) = \int_{\mathbb{R}^2} f(z)\, K^\gamma_{\alpha,\beta}(\zeta, z)\, dz \tag{8}$$

where

$$K^\gamma_{\alpha,\beta}(\zeta, z) = d(\gamma)\, e^{-i\left(a(\gamma)(|z|^2+|\zeta|^2+|m|^2) + z\cdot A(m-\zeta) + \zeta\cdot(n - ma(\gamma))\right)} \tag{9}$$

In this case, for each parameter of the kernel we have

$$\gamma = \tfrac{\alpha+\beta}{2}, \quad \delta = \tfrac{\alpha+\beta}{2}, \quad a(\gamma) = \cot\tfrac{\alpha}{2},$$

$$b(\gamma,\delta) = \frac{\cos\delta}{\sin\gamma}, \quad c(\gamma,\delta) = \frac{\sin\delta}{\sin\gamma}, \quad d(\gamma) = \frac{ie^{-i\gamma}}{2\pi\sin\gamma},$$

$$M = \begin{pmatrix} b(\gamma,\delta) & c(\gamma,\delta) \\ -c(\gamma,\delta) & b(\gamma,\delta) \end{pmatrix}.$$

The inverse of the Coupled offset fractional Fourier Transform obtained from the above definition is as follows.

Definition 3.2. For each $M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) \in L^2(\mathbb{R}^2)$, the inverse of the Coupled offset fractional Fourier Transform is obtained.

$$f(z) = \int_{\mathbb{R}^2} M^{\gamma}_{\alpha,\beta}\{f\}(\zeta)\,\overline{K^{\gamma}_{\alpha,\beta}(\zeta,z)}\,d\zeta \tag{10}$$

For a particular case, if $\alpha = \beta$, we get

$$\gamma = \alpha, \quad \delta = 0, \quad a(\alpha) = \cot\alpha, \quad b(\alpha) = \frac{1}{\sin\alpha}, \quad m = n = 0,$$

$$c(\alpha) = 0, \quad d(\alpha) = \frac{1}{2\pi}(1 + i\cot\alpha), \quad A = \begin{pmatrix} \frac{1}{\sin\alpha} & 0 \\ 0 & \frac{1}{\sin\alpha} \end{pmatrix}.$$

By substituting the above parameters into Definition 3.1, we get

$$\begin{aligned}
M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) &= \int_{\mathbb{R}^2} d(\gamma) f(z) e^{-i\left(a(\gamma)(|z|^2+|\zeta|^2+|m|^2)+z\cdot A(m-\zeta)+\zeta\cdot(n-ma(\gamma))\right)}\,dz \\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(z)\,e^{-i\left(\frac{\cot\alpha}{2}(|z|^2+|\zeta|^2)-\begin{pmatrix} z_1 & z_2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sin\alpha} & 0 \\ 0 & \frac{1}{\sin\alpha} \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\right)}\,dz \\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(z)\,e^{-i\left(\frac{\cot\alpha}{2}(|z|^2+|\zeta|^2)-\frac{z_1\zeta_1}{\sin\alpha}-\frac{z_2\zeta_2}{\sin\alpha}\right)}\,dz. \\
&= \frac{1+i\cot\alpha}{2\pi}\int_{\mathbb{R}^2} f(z)\,\exp\left(-i\left(\tfrac{\cot\alpha}{2}(|z|^2+|\zeta|^2)-(z\cdot\zeta)\csc\alpha\right)\right)dz \\
&= e^{i\left(\alpha+\frac{\pi}{2}\right)}\,\sin\alpha\,\mathscr{F}_{\alpha}\{f\}(\zeta).
\end{aligned}$$

An example of the definition of the Coupled Offset Fractional Fourier Transform with a Gaussian function is given below.

Example 3.1. Given a function f as follows

$$f(z) = e^{-|z|^2}.$$

Find its the Coupled Offset Fractional Fourier Transform.

Solution. From the definition we have

$$\begin{aligned}
M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) &= \int_{\mathbb{R}^2} f(z)\,K^{\gamma}_{\alpha,\beta}(\zeta,z)\,dz \\
&= d(\gamma)\int_{\mathbb{R}^2} e^{-|z|^2}\,e^{-i\left(a(\gamma)(|z|^2+|\zeta|^2+|m|^2)+z\cdot A(m-\zeta)+\zeta\cdot(n-ma(\gamma))\right)}\,dz \\
&= d(\gamma)e^{-i\left(a(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))\right)}\int_{\mathbb{R}^2} e^{-|z|^2}e^{-ia(\gamma)|z|^2}e^{-i(z\cdot A(m-\zeta))}\,dz.
\end{aligned}$$

Further, we get

$$\begin{aligned}
M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) &= d(\gamma)e^{-i\left(a(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))\right)} \\
&\quad\times\int_{\mathbb{R}^2} e^{-(z_1^2+z_2^2)}e^{-ia(\gamma)(z_1^2+z_2^2)}e^{-i\begin{pmatrix} z_1 & z_2 \end{pmatrix}\begin{pmatrix} b(\gamma,\delta) & c(\gamma,\delta) \\ -c(\gamma,\delta) & b(\gamma,\delta) \end{pmatrix}\begin{pmatrix} m_1-\zeta_1 \\ m_2-\zeta_2 \end{pmatrix}}\,dz_1dz_2.
\end{aligned}$$

Thus,

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) = d(\gamma)\, e^{-i\left(a(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))\right)} \\ \times \int_{\mathbb{R}^2} e^{-(1+ia(\gamma))z_1^2 - i\zeta_1\left(b(\gamma,\delta)(m_1-y_1)+c(\gamma,\delta)(m_2-y_2)\right)}\, dz_1 \\ \times \int_{\mathbb{R}^2} e^{-(1+ia(\gamma))z_2^2 - i\zeta_2\left(-c(\gamma,\delta)(m_1-y_1)+b(\gamma,\delta)(m_2-y_2)\right)}\, dz_2. \end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}} e^{-\alpha z^2-\beta z}\, dz = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{\beta^2}{4\alpha}}, \quad \text{where } \Re(\alpha) > 0,$$

in the right segment of the above equation we obtain

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) = \frac{\pi}{1+ia(\gamma)}\, e^{-ia(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))} \\ \times e^{\frac{\left(b(\gamma,\delta)(m_1-y_1)+c(\gamma,\delta)(m_2-y_2)\right)^2}{4(1+ia(\gamma))}} \\ \times e^{\frac{\left(-c(\gamma,\delta)(m_1-y_1)+b(\gamma,\delta)(m_2-y_2)\right)^2}{4(1+ia(\gamma))}}. \end{aligned}$$

Here is the relation between the offset coupled fractional Fourier transform and the two-dimensional Fourier transform. From the definition of the offset coupled fraction Fourier Transform, we get

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) &= d(\gamma) \int_{\mathbb{R}^2} f(z) e^{-i\left(a(\gamma)(|z|^2+|m|^2)+2z\cdot A(m-\zeta)+\zeta\cdot(n-ma(\psi))\right)}\, dz \\ &= d(\gamma)\, e^{-ia(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))} \int_{\mathbb{R}^2} f(z)\, e^{-a(\gamma)|z|^2+z\cdot Am}\, e^{iz\cdot A\zeta}\, dz \end{aligned}$$

or can be simplified to

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f\}(\zeta) &= d(\gamma)\, e^{-ia(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))} \int_{\mathbb{R}^2} f_{\alpha}(z)\, e^{iz\cdot A\zeta}\, dz \\ &= d(\gamma)\, e^{-ia(\gamma)(|\zeta|^2+|m|^2)+\zeta\cdot(n-ma(\gamma))}\, \mathscr{F}\{f_{\alpha}\}(-A\zeta) \end{aligned}$$

where

$$f_{\alpha}(z) = f(z)\, e^{-i\left(a(\gamma)|z|^2+z\cdot Am\right)},$$

with 2-dimensional Fourier transform defined as

$$\mathscr{F}\{f\}(\zeta) = \int_{\mathbb{R}^2} f(z)\, e^{-iz\cdot\zeta}\, dz.$$

□

Now we provide the proof of Parseval formula for the COFRFT using the direct relationship between the FT and COFrFT.

Lemma 3.1. For all $f, g \in L^2(\mathbb{R}^2)$, the following relation holds:

$$4\pi^2 \sin^{4\gamma} \int_{\mathbb{R}^2} f(z)\overline{h(z)}\, dz = \int_{\mathbb{R}^2} M^{\gamma}_{\alpha,\beta}\{f\}(\zeta)\, \overline{M^{\gamma}_{\alpha,\beta}\{h\}(\zeta)}\, d\zeta. \tag{11}$$

Proof. Based on the Parseval's formula of the FT in equation (11), we have

$$\int_{\mathbb{R}^2} f(z)\overline{h(z)}\,dz = \int_{\mathbb{R}^2} \mathscr{F}\{f\}(\zeta)\,\overline{\mathscr{F}\{h\}(\zeta)}\,d\zeta. \tag{12}$$

Replacing f with $f_\alpha(z)$ and h with $h_\alpha(z)$ in both sides of equation (12), we see that

$$\int_{\mathbb{R}^2} f_\alpha(z)\overline{h_\alpha(z)}\,dz = \int_{\mathbb{R}^2} \mathscr{F}\{f_\alpha\}(\zeta)\,\overline{\mathscr{F}\{h_\alpha\}(\zeta)}\,d\zeta.$$

And application of the relationship between FT and COFrFT, we get

$$\int_{\mathbb{R}^2} f(z)\overline{h(z)}\,dz = \frac{|d(\gamma)|^2}{\sin^{2\gamma}} \int_{\mathbb{R}^2} \mathscr{F}\{f_\alpha\}(-A\zeta)\,\overline{\mathscr{F}\{h_\alpha\}(-A\zeta)}\,d\zeta.$$

Or, equivalenly which finishes the proof. □

Next, the convolution and correlation of the Coupled Offset Fractional Fourier Transform will be formed as follows.

3.2 Convolution

In this section, the convolution form of the Coupled offset fractional Fourier Transform will be shown.

Definition 3.3. Convolution operation on Coupled offset fractional Fourier Transform for $f,\ h\ \in\ L^2(R^2)$ is given by

$$(f*h)(z) = \int_{\mathbb{R}^2} f(y)h(z-y)\,e^{-i\alpha(\gamma)|z-y|^2} \times e^{-i\alpha(\gamma)|z|^2} e^{iy\cdot Am}\,dy$$

From the definition, the convolution form of the Coupled offset fractional Fourier Transform is obtained which is poured into the following theorem.

Theorem 3.2. Given $f,h \in L^2(\mathbb{R}^2)$, the convolution of the offset coupled fractional Fourier transform of f and h is obtained.

$$M^\gamma_{\alpha,\beta}\{f*h\}(\zeta) = M^\gamma_{\alpha,\beta}\{h\}(\zeta)\,\mathscr{F}\{f\}(-A\zeta). \tag{3.6}$$

Proof. Based on the definition of Coupled offset fractional Fourier Transform, we get

$$\begin{aligned}
M^\gamma_{\alpha,\beta}\{f*h\}(\zeta) &= \int_{\mathbb{R}^2} \{f*h\}(z)\,K^\gamma_{\alpha,\beta}(\zeta,z)\,dz. \\
&= d(\gamma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)h(z-y)e^{-i\alpha(\gamma)|z-y|^2} e^{-i\alpha(\gamma)|z|^2} e^{iy\cdot Am}\,dz\,dy \\
&= d(\gamma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)h(z-y)e^{-i\alpha(\gamma)|z-y|^2} e^{iy\cdot Am}\,e^{-i\alpha(\gamma)(|z|^2+|z|^2+|m|^2)+z\cdot Am-Q+(x-m\alpha(\gamma))}\,dz\,dy \\
&= d(\gamma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)h(z-y)e^{-i\alpha(\gamma)|z-y|^2} e^{iy\cdot Am}\,e^{-i\alpha(\gamma)(|z|^2+|m|^2)+z\cdot Am-Q+(x-m\alpha(\gamma))}\,dz\,dy
\end{aligned}$$

By using the substitution integral by supposing the variable $x = z-y$, then we get

$$\begin{aligned}
M^\gamma_{\alpha,\beta}\{f*h\}(\zeta) &= d(\gamma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)h(x)e^{-i\alpha(\gamma)|x|^2} e^{iy\cdot Am}\,e^{-i\alpha(\gamma)(|x|^2+|m|^2)+(x+y)\cdot Am-Q+(x-m\alpha(\gamma))}\,dx\,dy \\
&= d(\gamma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)h(x)e^{iy\cdot Am}e^{-y\cdot Am-Q}\,e^{-i\alpha(\gamma)(|x|^2+|m|^2)+x\cdot Am-Q+(x-m\alpha(\gamma))}\,dx\,dy \\
&= d(\gamma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(y)h(x)e^{y\cdot A\zeta}\,e^{-i\alpha(\gamma)(|x|^2+|m|^2)+x\cdot Am-Q+(x-m\alpha(\gamma))}\,dx\,dy \\
&= d(\gamma)\int_{\mathbb{R}^2}\left[h(x)e^{-i\alpha(\gamma)(|x|^2+|m|^2)+x\cdot Am-Q+(x-m\alpha(\gamma))}\right] \\
&\qquad \times\left(\int_{\mathbb{R}^2} f(y)e^{y\cdot A\zeta}dy\right)dx \\
&= M^\gamma_{\alpha,\beta}\{h\}(\zeta)\,\mathscr{F}\{f\}(-A\zeta).
\end{aligned}$$

which finishes the proof. □

The correlations will be formed on the Coupled offset fractional Fourier Transform. First, the definition of the operation on the correlation is given and from this definition, the correlation on the Coupled offset fractional Fourier transform will be formed and we proof alternative solustion for correlation theorem using the definition of the convolution.

3.3 Correlation

In this section, the correlation form of the Coupled offset fractional Fourier Transform will be shown.

Definition 3.4. The correlation operation on the Coupled offset fractional Fourier Transform for $f,h \in L^2(\mathbb{R}^2)$ is given as follows

$$(f \circ h)(z) = \int_{\mathbb{R}^2} \overline{f(y)}\, h(z+y)\, e^{-i\alpha(\gamma)|z+y|^2} e^{-i\alpha(\gamma)|z|^2} e^{-iy\cdot Am}\, dy.$$

Based on the definition, the correlation form of the Coupled offset fractional Fourier Transform is obtained which is poured into the following theorem.

Theorem 3.3. Given $f,h \in L^2(\mathbb{R}^2)$ then the correlation of the offset coupled fractional Fourier transform of f and h is obtained.

$$M^{\gamma}_{\alpha,\beta}\{f \circ h\}(\zeta) = M^{\gamma}_{\alpha,\beta}\{h\}(\zeta)\, \mathscr{F}\{\overline{f}\}(-A\zeta)$$

Proof. Based on the definition of Coupled offset fractional Fourier Transform, we get

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f \circ h\}(\zeta) &= \int_{\mathbb{R}} (f \circ h)(z)\, K^{\gamma}_{\alpha,\beta}(\zeta,z)\, dz \\ &= d(\gamma) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{f(y)}\, h(z+y)\, e^{-i\alpha(\gamma)|z+y|^2}\, e^{-i\alpha(\gamma)|z|^2}\, e^{-iy\cdot Am}\, dz\, dy \\ &= d(\gamma) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{f(y)}\, h(z+y)\, e^{-i\alpha(\gamma)|z+y|^2}\, e^{-iy\cdot Am}\, e^{-i\alpha(\gamma)(|z|^2+|m|^2)+z\cdot Am-Q+(z-m\alpha(\gamma))}\, dz\, dy \end{aligned}$$

By using the substitution integral by assuming the variable $x = z+y$, then we get

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f \circ h\}(\zeta) &= d(\gamma) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{f(y)}\, h(x)\, e^{-i\alpha(\gamma)|x|^2}\, e^{-iy\cdot Am} \\ &\quad \times e^{-i\alpha(\gamma)(|x|^2+|m|^2)+(x-y)\cdot Am-Q+(x-m\alpha(\gamma))}\, dy\, dx \\ &= d(\gamma) \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \overline{f(y)}\, h(x)\, e^{-i\alpha(\gamma)(|x|^2+|m|^2)+x\cdot Am-Q}\, dy \right] \\ &\quad \times \zeta \cdot (n - m\alpha(\gamma))\, e^{-iy\cdot Am} e^{-iy\cdot Am-Q}\, dy\, dx \end{aligned}$$

Therefore,

$$\begin{aligned} M^{\gamma}_{\alpha,\beta}\{f \circ h\}(\zeta) &= d(\gamma) \int_{\mathbb{R}^2} h(x)\, e^{-i\alpha(\gamma)(|x|^2+|m|^2)+x\cdot Am-Q+(x-m\alpha(\gamma))} \\ &\quad \times \left(\int_{\mathbb{R}^2} \overline{f(y)}\, e^{y\cdot A\zeta}\, dy \right) dx \\ &= M^{\gamma}_{\alpha,\beta}\{h\}(\zeta)\, \mathscr{F}\{\overline{f}\}(-A\zeta). \end{aligned}$$

Which finishes the proof. □

Further proof of the correlation theorem for the coupled offset fractional Fourier transform using the relation between the definition of convolution and the definition of correlation will be given. Alternative proof of the correlation theorem

Substitution $y=-x$ and suppose $f(-z)=g(z)$ we get

$$\begin{aligned}(f\circ h)(z) &= \int_{\mathbb{R}^2} \overline{f(-x)}\, h(z-x)\, e^{-i\alpha(\gamma)|z-x|^2}\, e^{-i\alpha(\gamma)|z|^2}\, e^{ix\cdot Am}\, dx \\ &= \int_{\mathbb{R}^2} \overline{g(z)}\, h(z-x)\, e^{-i\alpha(\gamma)|z-x|^2}\, dx \\ &= (\overline{g}*h)(z)\end{aligned}$$

By using the equation (3.15) we get

$$M'_{\alpha,\beta}(\overline{g}*h)(\zeta) = M'_{\alpha,\beta}\{h\}(\zeta)\, \mathscr{F}[\overline{g}](-A\zeta). \tag{13}$$

Now consider that

$$\begin{aligned}\mathscr{F}[\overline{g}](-A\zeta) &= \int_{\mathbb{R}^2} \overline{g(z)}\, e^{iz\cdot A\zeta}\, dz \\ &= \int_{\mathbb{R}^2} \overline{f(-z)}\, e^{iz\cdot A\zeta}\, dz \\ &= \int_{\mathbb{R}^2} \overline{f(x)}\, e^{-ix\cdot A\zeta}\, dx \\ &= \int_{\mathbb{R}^2} f(x)\, e^{-ix\cdot A\zeta}\, dx \\ &= \mathscr{F}[f](-A\zeta).\end{aligned}$$

Substitute the equation above into equation (3.15), we get

$$M'_{\alpha,\beta}(\overline{g}*h)(\zeta) = M'_{\alpha,\beta}\{h\}(\zeta)\, \overline{\mathscr{F}[f](-A\zeta)}. \tag{14}$$

And from equation (3.14), then left hand side equation we get

$$M'_{\alpha,\beta}\{f\circ h\}(\zeta) = M'_{\alpha,\beta}\{h\}(\zeta)\, \overline{\mathscr{F}[f](-A\zeta)}.$$

which finishes the proof.

4. CONCLUSION

In this work, Coupled offset fractional Fourier transform is introduced. The convolution theorem and correlation associated with this transform are derived in detail. We find that the convolution theorem is equivalent to the simple multiplication of the Coupled offset fractional Fourier transform and the two-dimensional Fourier transform. In the future, we will concentrate on investigating the properties and uncertainty principles of these transforms, which proofs are more complicated, and discuss some applications, such as frequency filter analysis and etc.

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