Jurnal Matematika dan Statistika serta Aplikasinya, Vol. 13, No. 2, 220-223, 2025
DOI:https://doi.org/10.24252/msa.v13i2.65808

On Trees with Locating-Chromatic Number and Dominating Partition Dimension Equal to Three

Muhammad Ridwan[1]*, Wahyuni Abidin[2]
[1,2]*Mathematics Study Program, Universitas Islam Negeri Alauddin Makassar, Indonesia*
***Corresponding author**: muhammadridwan@uin-alauddin.ac.id

***Submission date**: 05 July 2025, **Revision**: 13 August 2025, **Accepted**: 24 December 2025

ABSTRACT
Several graph parameters are defined through vertex partitions and distance representations. Two such parameters are the locating-chromatic number $\chi_L(G)$ and the dominating partition dimension $\eta_p(G)$. The locating-chromatic number requires that vertices are distinguished by their distances to color classes, while the dominating partition dimension additionally imposes a domination condition, namely that every vertex must be adjacent to at least one partition class. It is known that for every connected graph G, $\beta_p(G) \leq \eta_p(G) \leq \chi_L(G)$, where $\beta_p(G)$ denotes the partition dimension. However, the conditions under which equality holds are not yet fully understood. In this paper, we focus on trees T and show that the equality $\chi_L(T) = \eta_p(T)$ occurs exactly at value three. More precisely, we prove that a tree T satisfies $\chi_L(T) = 3$ if and only if $\eta_p(T) = 3$. The proof is structural and is based on an analysis of leaf configurations and strong support vertices. In particular, we establish that the presence of two distinct strong support vertices forces $\eta_p(T) \geq 4$, preventing equality at value three. As a consequence, the class of trees with locating-chromatic number three coincides exactly with the class of trees whose dominating partition dimension equals three.

KEYWORDS
Locating-chromatic number, dominating partition dimension, partition dimension, tree.

1. INTRODUCTION

All graphs considered in this paper are finite, simple, and connected. Let $G = (V,E)$ be a connected graph. For a vertex $v \in V(G)$ and a nonempty subset $S \subseteq V(G)$, the distance from v to S is defined as

$$d(v,S) = \min\{d(v,x) : x \in S\}. \tag{1}$$

One way to distinguish vertices in a graph using distance information is through vertex partitions. Chartrand, Salehi, and Zhang [1] introduced the notion of a resolving partition as an analogue of the metric dimension defined in terms of vertex partitions. A vertex partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is called a resolving partition if every pair of distinct vertices $u, v \in V(G)$ has distinct distance representations

$$(d(u,S_1), d(u,S_2), \ldots, d(u,S_k)) \neq (d(v,S_1), d(v,S_2), \ldots, d(v,S_k)). \tag{2}$$

The minimum cardinality of a resolving partition of G is called the partition dimension and is denoted by $\beta_p(G)$.

Distance representations can also be used in vertex colorings by taking color classes as reference sets. This idea led Chartrand, Erwin, Henning, Slater, and Zhang [2] to introduce the locating-chromatic number as a refinement of proper coloring. Let $c : V(G) \to \{1,2,\ldots,k\}$ be a proper coloring of G with color classes $V_1, V_2, \ldots, V_k$. The color code of a vertex $v \in V(G)$ is defined as

$$(d(v,V_1), d(v,V_2), \ldots, d(v,V_k)). \tag{3}$$

If all vertices have distinct color codes, then c is called a locating-coloring, and the minimum such k is called the locating-chromatic number, denoted by $\chi_L(G)$. For trees, this parameter is known to be highly sensitive to the distribution of leaves. In particular, Syofyan, Baskoro, and Assiyatun [3] showed that the value of $\chi_L(T)$ is determined by the number of leaves adjacent to a single stem, while Baskoro and Asmiati [4] provided a complete characterization of all trees with locating-chromatic number three.

While partition dimension and locating-chromatic number are both defined in terms of distance representations, domination is not considered in either parameter. Motivated by the idea of combining domination and resolving concepts, Hernando, Mora, and Pelayo [5] introduced resolving dominating partitions and defined the dominating partition dimension. A vertex partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is said to be dominating if for every vertex $v \in V(G)$ there exists a part $S_i \in \Pi$ such that $d(v, S_i) = 1$. If a partition is both resolving and dominating, then it is called a resolving dominating partition, and the minimum cardinality of such a partition is called the dominating partition dimension, denoted by $\eta_p(G)$. They proved that for every connected graph G,

$$\beta_p(G) \leq \eta_p(G) \leq \beta_p(G) + 1. \tag{4}$$

Recently, Ridwan, Assiyatun, and Baskoro [6] strengthened this relationship by showing that

$$\beta_p(G) \leq \eta_p(G) \leq \chi_L(G). \tag{5}$$

In this direction, we investigate whether imposing domination constraints on resolving partitions preserves the structural properties of trees with small locating-chromatic number. Despite the complete structural understanding of trees with locating-chromatic number three, it is not immediate whether the same structural restrictions remain valid when domination is required. Motivated by this question, we focus on trees and prove that the equality $\chi_L(T) = \eta_p(T)$ occurs exactly at value three. Our results show that the same leaf-based obstructions governing locating-colorings also determine the dominating partition dimension, revealing a precise structural correspondence between the two parameters for trees.

2. MAIN RESULTS

Throughout this section, let T be a tree of order at least three. Recall that a vertex is called a *support vertex* if it is adjacent to at least one leaf, and it is called a *strong support vertex* if it is adjacent to at least two distinct leaves. We first establish a general lower bound for the dominating partition dimension of trees, and then derive structural restrictions imposed by resolving and domination requirements on vertex partitions. These restrictions lead to the main equivalence result of this paper.

Lemma 2.1. For any tree T of order at least three,

$$\eta_p(T) \geq 3. \tag{6}$$

Proof. By [1], every nontrivial tree satisfies $\beta_p(T) \geq 3$. Since $\eta_p(G) \geq \beta_p(G)$ for every connected graph G [5], the inequality $\eta_p(T) \geq 3$ follows. □

The next lemma isolates a domination constraint that is particularly restrictive at leaves. It will be used to control the possible placement of leaves and their support vertices in dominating partitions.

Lemma 2.2. Let G be a connected graph and let x be a leaf with unique neighbor s. If $\Pi = \{S_1, \ldots, S_k\}$ is a dominating partition of G, then x and s do not belong to the same part of Π.

Proof. Suppose, to the contrary, that $x, s \in S_i$ for some i. Since Π is dominating, there exists a part S_j such that $d(x, S_j) = 1$. Because x is a leaf and $N(x) = \{s\}$, the only vertices at distance 1 from x are those in $\{s\}$. Hence $s \in S_j$. But $s \in S_i$, so $S_j = S_i$, which contradicts $d(x, S_i) = 0$. Therefore x and s must lie in different parts. □

We now turn to the resolving requirement. In a tree, two leaves adjacent to the same support vertex are indistinguishable from the rest of the graph unless they are placed in different parts.

Lemma 2.3. Let T be a tree and let u be a support vertex adjacent to two distinct leaves u_1 and u_2. Then u_1 and u_2 must belong to different parts of any resolving partition of T.

Proof. For any vertex $x \in V(T) \setminus \{u_1, u_2\}$, the unique paths from u_1 to x and from u_2 to x both pass through u. Hence $d(u_1,x) = d(u_2,x)$. If u_1 and u_2 were contained in the same part of a partition, then their distance representations would coincide, contradicting the resolving property. □

Combining Lemma 2.2 and Lemma 2.3 yields a strong obstruction when such leaf configurations occur at two different locations in the tree. This obstruction forces the dominating partition dimension to be at least four.

Lemma 2.4. If a tree T contains two distinct strong support vertices, then

$$\eta_p(T) \geq 4. \tag{7}$$

Proof. Assume to the contrary that $\eta_p(T) = 3$. Then T admits a resolving dominating partition $\Pi = \{S_1, S_2, S_3\}$. Let u and v be two distinct strong support vertices of T, and let u_1, u_2 and v_1, v_2 be leaves adjacent to u and v, respectively.

Step 1: Leaf separation at u. By Lemma 2.3, the leaves u_1 and u_2 belong to different parts of Π. Relabel the parts if necessary so that $u_1 \in S_1$ and $u_2 \in S_2$.

Step 2: Domination forces $u \in S_3$. By Lemma 2.2, $u \notin S_1$ and $u \notin S_2$, and hence $u \in S_3$.

Step 3: Leaf separation and domination at v. Applying Lemma 2.2 to the leaves v_1 and v_2 shows that neither v_1 nor v_2 lies in S_3. Moreover, Lemma 2.3 implies that v_1 and v_2 belong to different parts. Thus, after possibly swapping S_1 and S_2, we may assume $v_1 \in S_1$ and $v_2 \in S_2$, and again Lemma 2.2 forces $v \in S_3$.

Step 4: Resolving fails. Comparing the distance representations of u_1 and v_1 with respect to Π, we obtain

$$(d(u_1,S_1), d(u_1,S_2), d(u_1,S_3)) = (0,2,1) = (d(v_1,S_1), d(v_1,S_2), d(v_1,S_3)),$$

which contradicts the resolving property of Π. Therefore, $\eta_p(T) \geq 4$. □

We are now ready to relate the dominating partition dimension to the locating-chromatic number for trees.

Theorem 2.5. Let T be a tree of order at least three. Then

$$\chi_L(T) = 3 \quad \text{if and only if} \quad \eta_p(T) = 3. \tag{8}$$

Proof. ($\Rightarrow$) Suppose $\chi_L(T) = 3$. By [6], we have $\eta_p(T) \leq \chi_L(T) = 3$. Together with Lemma 2.1, this implies $\eta_p(T) = 3$.

($\Leftarrow$) Suppose $\eta_p(T) = 3$. By Lemma 2.4, the tree T contains at most one strong support vertex. On the other hand, Baskoro and Asmiati [4] showed that if a tree contains a vertex adjacent to at least two leaves, then its locating-chromatic number is at least four. Hence $\chi_L(T) \leq 3$. Since every tree of order at least three satisfies $\chi_L(T) \geq 3$, it follows that $\chi_L(T) = 3$. □

3. CONCLUSION

We have shown that for trees of order at least three, the equality

$$\chi_L(T) = \eta_p(T) = 3$$

holds precisely for the class of trees characterized in [4]. The structural obstruction is the presence of multiple strong support vertices, which forces $\eta_p(T) \geq 4$.

This result clarifies the interaction between locating-coloring and domination-based partition parameters in trees and suggests that nontrivial divergence between these parameters may occur only for larger values.

REFERENCES

[1] G. Chartrand, E. Salehi, and P. Zhang, "The partition dimension of a graph," *Aequationes Mathematicae*, vol. 1, no. 59, pp. 45–54, 2000.

[2] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater, and P. Zhang, "The locating-chromatic number of a graph," *Bulletin of the ICA*, vol. 36, pp. 89–101, 2002.

[3] D. K. Syofyan, E. T. Baskoro, and H. Assiyatun, "Trees with certain locating-chromatic number," *J. Math. Fund. Sci*, vol. 48, no. 1, pp. 39–47, 2016.

[4] E. T. Baskoro and A. Asmiati, "Characterizing trees with locating-chromatic number 3," *Electronic Journal of Graph Theory and Applications*, vol. 1, no. 2, pp. 109–117, 2013.

[5] C. Hernando, M. Mora, and I. M. Pelayo, "Resolving dominating partitions in graphs," *Discrete Applied Mathematics*, vol. 266, pp. 237–251, 2019.

[6] M. Ridwan, H. Assiyatun, and E. T. Baskoro, "The dominating partition dimension and locating-chromatic number of graphs," *Electronic Journal of Graph Theory and Applications*, vol. 2, no. 11, pp. 455–465, 2023.