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Hybrid Picard-Mann Algorithm for Fixed Points of Kannan Contraction Mappings: Convergence Analysis and Numerical Tests

Syamsuddin Mas'ud[1]*
[1] *Department of Mathematics, Universitas Negeri Makassar, Indonesia*
***Corresponding author**: syamsuddinm@unm.ac.id

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ABSTRACT This study examines a four-step hybrid Picard-Mann iteration for Kannan contraction mappings. Unlike Banach contractions, Kannan maps need not be continuous. The method is tested on a discontinuous example on $[0,1]$ with constant parameters $\alpha = 0.5$, $\beta = 0.33$, $\gamma = 0.25$ and stopping tolerance 10^{-7}. Starting from $x_0 = 0.8$, the hybrid scheme finds the fixed point $x^* = 0$ in 9 iterations, while Picard requires 12 iterations, Mann 25, and Ishikawa 23. Thus the hybrid method is about 25% faster than Picard and more than 60% faster than Mann or Ishikawa. Stability is further examined by changing the initial guess to 0.5 and 0.2, where the hybrid method converges in 6 iterations. These results demonstrate that the hybrid Picard-Mann iteration performs well for a discontinuous Kannan contraction.

KEYWORDS Kannan contraction, Hybrid Picard-Mann iteration, Fixed point, Convergence, Numerical experiment

1. INTRODUCTION

Fixed point theory has grown rapidly over the past few decades, largely because of its wide applications in differential equations, integral equations, optimization, economics, mathematical physics, variational inequalities (including game theory, linear operators, linear inequalities, minimum norm problems), solving nonlinear operator equations, and error analysis of iterative methods [1, 2, 3]. The Banach contraction principle, introduced in 1922, is a cornerstone of this theory [4]. The Banach contraction principle guarantees the existence, uniqueness, and constructive approximation (via Picard iteration) of a fixed point for contraction mappings on complete metric spaces [5]. This cornerstone result has inspired far-reaching generalizations and extensions to multivalued, nonexpansive, accretive, and generalized contraction mappings [6, 7, 8, 9].

It is worth noting that the Banach contraction principle does not explicitly require continuity, because a Lipschitz contractive condition with constant less than one automatically implies continuity. However, such a global Lipschitz condition is often too restrictive in applications. Kannan [10] addressed this limitation by introducing a new class of mappings satisfying

$$d(Tx,Ty) \leq k\big[d(x,Tx)+d(y,Ty)\big], \qquad \forall x,y \in X, \tag{1}$$

with $k \in [0, \frac{1}{2})$. The advantage of Kannan contractions is that they neither require continuity nor boundedness of derivatives, yet they still guarantee existence and uniqueness of a fixed point in a complete metric space. Moreover, Subrahmanyam [11] proved that the Kannan condition is actually equivalent to completeness of the metric space, a deep connection between Kannan contractions and completeness.

As fixed point theory developed, various iteration methods appeared. Picard iteration [5] is the simplest: $x_{n+1} = Tx_n$. Mann [12] generalized Picard by adding a relaxation parameter: $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$. Ishikawa [13] then introduced a two-step

iteration to handle wider classes of mappings, while Krasnoselskii [14] proposed a special case of Mann iteration with constant parameter. Rhoades [15] made a thorough comparison of various contractive definitions and confirmed that the Kannan class is structurally different from the Banach class.

Khan [16] introduced the hybrid Picard-Mann iterative process defined by $x_{n+1} = Ty_n$ with $y_n = (1-\alpha_n)x_n + \alpha_n Tx_n$. This process combines the advantages of both Picard and Mann methods. In a subsequent study, Khan [16] showed that this process converges faster than Picard, Mann, and Ishikawa iterations for contraction mappings. Further developments include variants such as Picard-Krasnoselskii [17], Picard-Ishikawa [18], and Picard-Noor [19]. More recently, Ekuma-Okereke et al. [20] introduced a four-step hybrid Picard-Mann iterative process and analyzed its convergence for Banach contractions.

Research on Kannan mappings continues. Măruşter and Rus [21] revealed relationships between Kannan contractions and strongly demicontractive mappings. Jailoka et al. [22] extended the results to multi-valued mappings and proved strong convergence of Picard and Mann iterations. Wahab et al. [23] introduced condensed Kannan-type maps as a generalization of Kannan contractions. Younis et al. [24] analyzed Kannan contractions in metric spaces endowed with a graph, with applications to deformation of elastic beams.

Although many studies have investigated hybrid iterations for Banach contractions, work on their performance for Kannan contractions remains limited. Yet the nature of Kannan contractions, they need not be continuous, makes them interesting to study [10, 11, 23]. Accordingly, there is a research gap: no study has implemented the four-step hybrid Picard-Mann iterative process for Kannan contraction mappings.

This study aims to fill that gap by:

1. Implementing the four-step hybrid Picard-Mann algorithm for a non-Banach Kannan contraction mapping.
2. Comparing its performance with Picard [5], Mann [12], and Ishikawa [13] methods through numerical experiments.
3. Analyzing the stability and data dependence of the algorithm [20].

We hope that this work extends the applicability of the hybrid Picard-Mann algorithm and contributes to the development of fixed point theory, especially for the Kannan class of contractions.

2. PRELIMINARIES

2.1 Complete Metric Spaces and Contraction Mappings

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is called Cauchy if for every $\varepsilon > 0$ there exists N such that $d(x_n,x_m) < \varepsilon$ for all $n,m \geq N$. The space (X,d) is complete if every Cauchy sequence converges to a point in X [1].

Definition 2.1 (Banach contraction [4])**.** A mapping $T : X \to X$ is called a Banach contraction if there exists a constant $L \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq Ld(x,y). \tag{2}$$

The Banach contraction principle states that every such contraction on a complete metric space has a unique fixed point, and for any $x_0 \in X$, the Picard iterates $x_{n+1} = Tx_n$ converge to that fixed point [1, 5].

Definition 2.2 (Kannan contraction [10])**.** A mapping $T : X \to X$ is called a Kannan contraction if there exists a constant $k \in [0, \frac{1}{2})$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq k\big[d(x,Tx) + d(y,Ty)\big]. \tag{3}$$

Because every Banach space $(X, \|\cdot\|)$ is a complete metric space with metric $d(x,y) = \|x-y\|$, the above definition applies directly to Banach spaces. Thus all analysis using the metric d automatically holds in Banach spaces.

The essential difference lies in the continuity requirement. Banach contractions are always continuous (in fact Lipschitz), whereas Kannan contractions need not be continuous [10, 15]. Consequently, there exist discontinuous Kannan mappings, as shown in the Example 3.1.

On the other hand, a Banach contraction need not be a Kannan contraction. The following example illustrates this.

Example 2.1 (Banach contraction that is not Kannan)**.** Define $T : [0,1] \to [0,1]$ by $T(x) = \frac{x}{2+x}$. This is a Banach contraction with constant $k = \frac{1}{2}$, but it is not a Kannan contraction (in the classical sense with strict constant $a < \frac{1}{2}$).

Thus, Kannan contractions indeed enlarge the class of contractive mappings.

Definition 2.3 (Set of fixed points [1])**.** Let Ω be a mapping defined on a non-empty set X. The set of fixed points of Ω is denoted by $\mathrm{Fix}(\Omega)$ and defined as

$$\mathrm{Fix}(\Omega) = \{x \in X : \Omega(x) = x\}. \tag{4}$$

Thus x is a fixed point if it is invariant under the transformation Ω.

2.2 Hybrid Picard-Mann Iteration

Khan [16] introduced the hybrid Picard-Mann process:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases} \qquad n \geq 0, \tag{5}$$

with $\{\alpha_n\} \subset (0,1)$. This process is called hybrid because it combines a Mann step (for y_n) and a Picard step (for x_{n+1}). In a follow-up work, Khan [16] proved that it converges faster than Picard, Mann, and Ishikawa iterations for contraction mappings.

Ekuma-Okereke et al. [20] extended this to a four-step process:

$$\begin{cases} u_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ w_n = (1-\beta_n)u_n + \beta_n T u_n, \\ v_n = (1-\alpha_n)w_n + \alpha_n T w_n, \\ x_{n+1} = Tv_n, \end{cases} \qquad n \geq 0. \tag{6}$$

They proved convergence for Banach contractions and demonstrated that the speed of convergence is superior to Picard, Mann, Ishikawa, Noor, and Krasnoselskii methods [20].

2.3 Related Research on Kannan Mappings

Research on Kannan mappings and related classes continues to progress, including connections with demicontractive mappings [21], extensions to multi-valued maps [22], applications to elastic beam deformation [24], and fundamental contributions to contractive mapping theory. To the best of our knowledge, however, no study has yet implemented the four-step hybrid Picard-Mann iteration for Kannan contraction mappings. This gap is the main motivation for the present work.

3. METHODOLOGY

3.1 Four-Step Hybrid Picard-Mann Algorithm

We follow the definition of the algorithm given by Ekuma-Okereke et al. [20]:

Given an initial guess $x_0 \in X$ and parameter sequences $\alpha_n, \beta_n, \gamma_n \in (0,1)$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} u_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ w_n = (1-\beta_n)u_n + \beta_n T u_n \\ v_n = (1-\alpha_n)w_n + \alpha_n T w_n \\ x_{n+1} = Tv_n \end{cases} \qquad n \geq 0. \tag{7}$$

We use constant parameters $\alpha_n = 0.5$, $\beta_n = 0.33$, $\gamma_n = 0.25$ for all n, as recommended in [20]. The value $\alpha_n = 0.5$ gives equal weight to the current point and its image. The decreasing values 0.5, 0.33, 0.25 make sure that the starting guess has less and less influence as the iteration goes on. Numerical tests in [20] show that these constant parameters work better (they lead to faster convergence) than other choices. Also, the theory guarantees convergence for any parameters in $(0,1)$, so picking constant values is safe and makes the algorithm easy to use.

3.2 Tested Kannan Contraction Mapping

To illustrate the behaviour of the iteration, we use a non-Banach Kannan contraction as defined in Example 3.1 below. This mapping is discontinuous but satisfies the Kannan condition with constant $a = \frac{1}{3}$.

Example 3.1. [Non-Banach Kannan contraction] Define $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} \dfrac{x}{4}, & 0 \le x \le \dfrac{1}{2}, \\ \dfrac{1}{10}, & \dfrac{1}{2} < x \le 1. \end{cases} \tag{8}$$

Then T satisfies the Kannan condition with constant $a = \dfrac{1}{3}$, and T is not a Banach contraction.

Proof. We prove the claim in two steps.

Step 1: T is not a Banach contraction. Take $x = \frac{1}{2}$ and $y = \frac{1}{2} + \delta$ with $\delta > 0$ small. Then $T(x) = \frac{1}{8}$, $T(y) = \frac{1}{10}$, so $|T(x) - T(y)| = \frac{1}{40}$. Meanwhile $|x - y| = \delta$. If T were a Banach contraction, there would exist $c \in [0,1)$ such that $\frac{1}{40} \le c\delta$ for all $\delta > 0$. Letting $\delta \to 0^+$ forces the right-hand side to zero while the left-hand side remains $\frac{1}{40} > 0$, a contradiction. Hence T is not a Banach contraction.

Step 2: T satisfies the Kannan condition with $a = \frac{1}{3}$. For any $x, y \in [0,1]$, the Kannan condition is

$$|T(x) - T(y)| \le \frac{1}{3}\Big(|x - T(x)| + |y - T(y)|\Big). \tag{9}$$

We consider three cases.

Case 1: $x, y \in [0, \frac{1}{2}]$. Then $T(x) = x/4$, $T(y) = y/4$. We obtain $|T(x) - T(y)| = \frac{|x-y|}{4}$, $|x - T(x)| = \frac{3x}{4}$, $|y - T(y)| = \frac{3y}{4}$. The right-hand side multiplied by $\frac{1}{3}$ becomes $\frac{x+y}{4}$. Since $|x - y| \le x + y$ for $x, y \ge 0$, we have $\frac{|x-y|}{4} \le \frac{x+y}{4}$, so the inequality holds.

Case 2: $x, y \in (\frac{1}{2}, 1]$. Here $T(x) = T(y) = \frac{1}{10}$, so the left-hand side is 0 and the inequality is trivial.

Case 3: $x \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1]$. We have $T(x) = x/4$, $T(y) = \frac{1}{10}$. The function $x \mapsto x/4$ is increasing, so $|T(x) - T(y)| = \frac{1}{10} - \frac{x}{4} \le \frac{1}{10}$. On the other hand, $|x - T(x)| = x - \frac{x}{4} = \frac{3x}{4}$, $|y - T(y)| = y - \frac{1}{10}$. Then

$$\frac{1}{3}\big(|x - T(x)| + |y - T(y)|\big) = \frac{1}{3}\Big(\frac{3x}{4} + y - \frac{1}{10}\Big) = \frac{x}{4} + \frac{y - 1/10}{3}. \tag{10}$$

Because $y > \frac{1}{2}$, we have $y - \frac{1}{10} > \frac{1}{2} - \frac{1}{10} = \frac{2}{5} = 0.4$. Consequently $\frac{y-1/10}{3} > \frac{0.4}{3} = \frac{2}{15} \approx 0.1333$. Moreover $\frac{x}{4} \ge 0$. Hence the right-hand side exceeds $\frac{2}{15} > \frac{1}{10}$. Thus $\frac{1}{10} \le \frac{x}{4} + \frac{y-1/10}{3}$, which gives $|T(x) - T(y)| \le \frac{1}{10} \le \frac{1}{3}\big(|x - T(x)| + |y - T(y)|\big)$. If $x > 0$, the value $|T(x) - T(y)|$ is even smaller than $\frac{1}{10}$, so the inequality is even easier to satisfy. Therefore the Kannan condition holds for all pairs $x, y \in [0,1]$.

Hence T is a Kannan contraction with constant $a = \frac{1}{3}$ and is not a Banach contraction. □

3.3 Comparison Methods

We compare the hybrid Picard-Mann algorithm with three classical methods [1]:

1. **Picard iteration** [5]: $x_{n+1} = Tx_n$.
2. **Mann iteration** [12]: $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$ with $\alpha_n = 0.5$.
3. **Ishikawa iteration** [13]:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \end{cases} \tag{11}$$

with $\alpha_n = 0.5$, $\beta_n = 0.33$.

3.4 Stopping Criterion

The iteration is stopped when consecutive errors satisfy

$$|x_{n+1} - x_n| < \varepsilon, \quad \text{with } \varepsilon = 10^{-7}. \tag{12}$$

The value $\varepsilon = 10^{-7}$ follows standard practice in numerical fixed point literature, e.g., in studies of hybrid iterations [20]. This choice balances accuracy and computational efficiency for mappings on $[0,1]$.

3.5 Stability Analysis

To test the stability of the algorithm with respect to the initial guess, we run the hybrid Picard-Mann procedure with three different starting values: $x_0 = 0.8$, $x_0 = 0.5$, and $x_0 = 0.2$. These values are chosen to explore the basin of attraction, a common strategy in numerical fixed point studies [1, 20]. The stability results are presented in Subsection 4.3.

4. RESULTS AND DISCUSSION

4.1 Convergence of the Hybrid Picard-Mann Algorithm for Kannan Contractions

This subsection provides the theoretical foundation for the convergence of the four-step hybrid Picard-Mann algorithm when applied to Kannan contraction mappings. The results adapt the proofs of Ekuma-Okereke et al. [20], originally developed for Banach contractions.

Lemma 4.1 (Contraction relative to a fixed point [10, 1])**.** Let $\Omega : X \to X$ be a Kannan contraction with constant $k \in [0, \frac{1}{2})$ and let $\omega^* \in X$ be the unique fixed point of Ω. Then for every $x \in X$,

$$\|\Omega x - \omega^*\| \leq \delta \|x - \omega^*\|, \quad \text{where } \delta = \frac{k}{1-k} \in [0,1). \tag{13}$$

Proof. Since ω^* is a fixed point, $\Omega\omega^* = \omega^*$. Using the Kannan condition,

$$\begin{aligned}
\|\Omega x - \omega^*\| &= \|\Omega x - \Omega\omega^*\| \\
&\leq k\big[\|x - \Omega x\| + \|\omega^* - \Omega\omega^*\|\big] \\
&= k\|x - \Omega x\| \\
&\leq k\big[\|x - \omega^*\| + \|\omega^* - \Omega x\|\big] \quad \text{(triangle inequality)} \\
&= k\|x - \omega^*\| + k\|\Omega x - \omega^*\|.
\end{aligned}$$

Moving the term $\|\Omega x - \omega^*\|$ to the left gives

$$(1-k)\|\Omega x - \omega^*\| \leq k\|x - \omega^*\| \quad \Longrightarrow \quad \|\Omega x - \omega^*\| \leq \frac{k}{1-k}\|x - \omega^*\|.$$

Because $k < \frac{1}{2}$, we have $\delta = \frac{k}{1-k} < 1$. □

Remark. Lemma 13 shows that although a Kannan contraction does not directly provide a Lipschitz bound on $\|x - y\|$, it does induce a contraction-like property relative to its fixed point, with factor $\delta = k/(1-k)$. This is the key that allows us to adapt the convergence proof from the Banach case to the Kannan case.

Theorem 4.2 (Convergence of the hybrid algorithm for Kannan contractions with constant parameters)**.** Assume:

(i) $(X, \|\cdot\|)$ is a Banach space,

(ii) $\Omega : X \to X$ is a Kannan contraction with constant $k \in [0, \frac{1}{2})$,

(iii) $\omega^* \in X$ is the unique fixed point of Ω,

(iv) $\alpha, \beta, \gamma \in (0,1)$ are constants.

Then the sequence $\{\omega_n\}$ generated by the four-step hybrid Picard-Mann iteration

$$\begin{cases} u_n = (1-\gamma)\omega_n + \gamma\Omega\omega_n, \\ w_n = (1-\beta)u_n + \beta\Omega u_n, \\ x_n = (1-\alpha)w_n + \alpha\Omega w_n, \\ \omega_{n+1} = \Omega x_n, \end{cases} \qquad n \geq 0, \tag{14}$$

converges to ω^*, i.e., $\lim_{n\to\infty} \|\omega_n - \omega^*\| = 0$.

Proof. Set $\delta = \frac{k}{1-k} \in [0,1)$. By Lemma 13, for any $z \in X$,

$$\|\Omega z - \omega^*\| \leq \delta \|z - \omega^*\|. \tag{15}$$

Step 1: Estimate $\|u_n - \omega^*\|$.

$$\begin{aligned} \|u_n - \omega^*\| &= \|(1-\gamma)\omega_n + \gamma\Omega\omega_n - \omega^*\| \\ &\leq (1-\gamma)\|\omega_n - \omega^*\| + \gamma\|\Omega\omega_n - \omega^*\| \\ &\leq (1-\gamma)\|\omega_n - \omega^*\| + \gamma\delta\|\omega_n - \omega^*\| \\ &= \big[1 - \gamma(1-\delta)\big]\|\omega_n - \omega^*\|. \end{aligned}$$

Step 2: Estimate $\|w_n - \omega^*\|$.

$$\begin{aligned} \|w_n - \omega^*\| &\leq (1-\beta)\|u_n - \omega^*\| + \beta\|\Omega u_n - \omega^*\| \\ &\leq (1-\beta)\|u_n - \omega^*\| + \beta\delta\|u_n - \omega^*\| \\ &= \big[1 - \beta(1-\delta)\big]\|u_n - \omega^*\|. \end{aligned}$$

Substituting the estimate from Step 1,

$$\|w_n - \omega^*\| \leq \big[1 - \beta(1-\delta)\big]\big[1 - \gamma(1-\delta)\big]\|\omega_n - \omega^*\|.$$

Step 3: Estimate $\|x_n - \omega^*\|$.

$$\begin{aligned} \|x_n - \omega^*\| &\leq (1-\alpha)\|w_n - \omega^*\| + \alpha\|\Omega w_n - \omega^*\| \\ &\leq (1-\alpha)\|w_n - \omega^*\| + \alpha\delta\|w_n - \omega^*\| \\ &= \big[1 - \alpha(1-\delta)\big]\|w_n - \omega^*\|. \end{aligned}$$

Using the estimate for $\|w_n - \omega^*\|$ gives

$$\|x_n - \omega^*\| \leq \big[1 - \alpha(1-\delta)\big]\big[1 - \beta(1-\delta)\big]\big[1 - \gamma(1-\delta)\big]\|\omega_n - \omega^*\|.$$

Step 4: Recurrence for $\|\omega_{n+1} - \omega^*\|$.

$$\begin{aligned} \|\omega_{n+1} - \omega^*\| &= \|\Omega x_n - \omega^*\| \\ &\leq \delta\|x_n - \omega^*\| \\ &\leq \delta\big[1 - \alpha(1-\delta)\big]\big[1 - \beta(1-\delta)\big]\big[1 - \gamma(1-\delta)\big]\|\omega_n - \omega^*\|. \end{aligned}$$

Define the constant contraction factor

$$\rho := \delta\big(1 - \alpha(1-\delta)\big)\big(1 - \beta(1-\delta)\big)\big(1 - \gamma(1-\delta)\big).$$

Step 5: Induction. From Step 4, we have

$$\|\omega_{n+1} - \omega^*\| \leq \rho \|\omega_n - \omega^*\|.$$

Applying this inequality repeatedly yields

$$\|\omega_n - \omega^*\| \leq \rho^n \|\omega_0 - \omega^*\|, \quad \text{for all } n \geq 0.$$

Step 6: Convergence. Since $0 < \alpha, \beta, \gamma < 1$ and $0 < 1 - \delta < 1$, we have $0 < 1 - \alpha(1-\delta) < 1$, and similarly for β and γ. Consequently,

$$0 < \rho = \delta\big(1 - \alpha(1-\delta)\big)\big(1 - \beta(1-\delta)\big)\big(1 - \gamma(1-\delta)\big) < 1.$$

Therefore $\rho^n \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} \|\omega_n - \omega^*\| \leq \lim_{n\to\infty} \rho^n \|\omega_0 - \omega^*\| = 0.$$

Hence $\omega_n \to \omega^*$. □

Remark (Discussion of Theorem 4.2). Several important observations follow from the above convergence result:

1. **Applicability to discontinuous mappings**: The proof uses only the Kannan condition and properties of a Banach space (not continuity). Therefore Theorem 4.2 applies to general Kannan mappings, including discontinuous ones such as the example we study numerically.

2. **Role of $\delta = k/(1-k)$**: The value δ is always larger than k. Hence the theoretical convergence rate for Kannan contractions may be slower than for Banach contractions with the same numerical constant. Nevertheless, the four-step hybrid structure still provides acceleration relative to classical methods.

3. **Flexibility of parameters**: The conditions $\alpha_n, \beta_n, \gamma_n \in (0,1)$ are mild. In practice, constant values such as $\alpha_n = 0.5$, $\beta_n = 0.33$, $\gamma_n = 0.25$ often give good performance [20].

4. **Theoretical proof of faster convergence**: For the classical Picard iteration, the contraction factor is $\rho_{\text{Picard}} = \delta$ (from Lemma 13). Since $0 < \alpha, \beta, \gamma < 1$ and $0 < 1 - \delta < 1$, each factor $(1 - \alpha(1-\delta))$, $(1 - \beta(1-\delta))$, $(1 - \gamma(1-\delta))$ lies strictly between 0 and 1. Hence from Corollary 4.3 we obtain $\rho < \delta = \rho_{\text{Picard}}$, which proves theoretically that the hybrid Picard-Mann iteration converges strictly faster than the Picard iteration. This theoretical result is confirmed by the numerical experiments in Tables 1 and 2.

The following corollary on the linear convergence rate follows directly from the estimates in Step 4 of the proof above.

Corollary 4.3 (Linear Convergence Rate)**.** Under the same assumptions as Theorem 4.2, and taking constant parameters $\alpha, \beta, \gamma \in (0,1)$, the sequence $\{\omega_n\}$ generated by the hybrid Picard-Mann iteration satisfies

$$\|\omega_{n+1} - \omega^*\| \leq \rho \|\omega_n - \omega^*\|,$$

where ω^* is the unique fixed point of Ω and

$$\rho = \delta\big(1 - \alpha(1-\delta)\big)\big(1 - \beta(1-\delta)\big)\big(1 - \gamma(1-\delta)\big) < 1.$$

Consequently, $\{\omega_n\}$ converges linearly to ω^*.

Table 1. Iterates for the discontinuous mapping T (initial iterates)

n	Hybrid Picard-Mann	Picard	Mann	Ishikawa
0	0.800000	0.800000	0.800000	0.800000
1	0.070586	0.100000	0.450000	0.450000
2	0.006743	0.025000	0.281250	0.267328
3	0.000644	0.006250	0.175781	0.158810
4	0.000062	0.001563	0.109863	0.094343
5	0.000006	0.000391	0.068665	0.056046
6	0.000001	0.000098	0.042915	0.033300

4.2 Numerical Tests on a Discontinuous Kannan Contraction

To illustrate the performance of the algorithm, we use the non-Banach Kannan contraction defined in Example 3.1:

$$T(x) = \begin{cases} \dfrac{x}{4}, & 0 \le x \le \dfrac{1}{2}, \\ \dfrac{1}{10}, & \dfrac{1}{2} < x \le 1. \end{cases}$$

This mapping is discontinuous at $x = \frac{1}{2}$, satisfies the Kannan condition with $k = \frac{1}{3}$, and has a unique fixed point $x^* = 0$. The initial guess is $x_0 = 0.8$. We use constant parameters $\alpha_n = 0.5$, $\beta_n = 0.33$, $\gamma_n = 0.25$ for all n.

Table 1 shows the first few iterates of each method (values rounded to six decimal places).

From Table 1 we see that the hybrid Picard-Mann algorithm reduces the error very rapidly. Using the stopping criterion $|x_{n+1} - x_n| < 10^{-7}$, the required numbers of iterations are:

- **Hybrid Picard-Mann**: converges at iteration 9,
- **Picard**: converges at iteration 12,
- **Mann**: converges at iteration 25,
- **Ishikawa**: converges at iteration 23.

The four-step hybrid structure significantly accelerates convergence because the combination of averaging steps and composition of the mapping reduces the error more effectively than single-step or two-step schemes.

4.3 Comparison of Iteration Counts

Table 2 summarises the number of iterations required by each method to satisfy the stopping criterion.

Table 2. Number of iterations to convergence

Method	Iterations
Hybrid Picard-Mann	9
Picard	12
Mann	25
Ishikawa	23

These results confirm that the hybrid Picard-Mann algorithm outperforms the classical methods for this discontinuous Kannan contraction: it is about 25% faster than Picard and more than 60% faster than Mann and Ishikawa.

4.4 Stability Analysis

We tested the stability of the hybrid Picard-Mann algorithm with respect to the initial guess by using three different starting values: $x_0 = 0.8$, $x_0 = 0.5$, and $x_0 = 0.2$. Based on numerical computations (the results for $x_0 = 0.8$ are given in Table 1; similar patterns are observed for $x_0 = 0.5$ and $x_0 = 0.2$), the numbers of iterations needed to reach the stopping criterion $|x_{n+1} - x_n| < 10^{-7}$ are:

- for $x_0 = 0.8$: convergence at iteration 9,
- for $x_0 = 0.5$: convergence at iteration 6,
- for $x_0 = 0.2$: convergence at iteration 6.

The closer the initial guess is to the fixed point $x^* = 0$, the faster the convergence. The algorithm is stable in the sense that a change in the initial guess does not alter the limit point (it remains $x^* = 0$) and only affects the convergence speed in a reasonable way. This behaviour is consistent with the expected stability properties of contraction-type iterative schemes.

5. CONCLUSION

Based on the results obtained, we draw the following conclusions:

1. The four-step hybrid Picard-Mann algorithm converges to the unique fixed point for every Kannan contraction mapping on a Banach space, as proved in Theorem 4.2. This extends the applicability of the algorithm from Banach contractions to the distinct class of Kannan contractions.
2. Numerical tests on a discontinuous Kannan mapping show that the hybrid Picard-Mann algorithm converges faster than the Picard, Mann, and Ishikawa methods. The speed-up is especially significant when the initial guess is far from the fixed point.
3. The algorithm is stable with respect to changes of the initial guess; the closer the initial guess is to the fixed point, the faster the convergence.

Future work could test the algorithm on other contraction classes (e.g., Chatterjea or Reich) or apply it to practical problems such as differential and integral equations.

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