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Characteristics of the Structure of Abelian Groups of Order (p^n) via Invariant Factors and Elementary Divisors

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ABSTRACT
Finite Abelian groups are one of the key objects in abstract algebra as their structure can be completely classified via the Fundamental Theorem of Finite Abelian Groups. This study aims to describe the classification of finite Abelian groups of order (p^n), explain the relationship between invariant factors and elementary divisors, and demonstrate the structural characteristics of the groups based on integer partitions. The research method employed was qualitative research using a literature review and theoretical analysis. The results show that every finite Abelian group of order (p^n) can be expressed as the direct product of cyclic groups of prime powers. The invariant factor form provides a hierarchical representation based on divisibility properties, whilst the elementary divisor form provides a representation based on elementary prime factors. The number of non-isomorphic Abelian groups of order (p^n) is equal to the number of partitions of (n).

KEYWORDS
Finite Abelian Group, Order (p^n), Invariant Factor, Elementary Divisor, Fundamental Theorem.

1. INTRODUCTION

A group is an algebraic structure with a binary operation that satisfies the axioms of closure, associativity, identity, and inverses. One of the most widely studied types of groups is the Abelian group, namely a group that satisfies the commutative property.

Finite Abelian groups are significant because their structure can be completely classified via the Fundamental Theorem of Finite Abelian Groups. This theorem states that every finite Abelian group can be expressed as a direct product of cyclic groups of prime-power order; there are two main forms of representation frequently used, namely invariant factor decomposition and elementary divisor decomposition.

The invariant factor representation provides a form of the group's structure based on a series of factors that satisfy the divisibility property. Meanwhile, the elementary divisor decomposition presents the group's structure as the direct product of cyclic groups of prime powers. These two approaches offer different yet equivalent perspectives on the structure of finite Abelian groups.

The study of groups of order (p^n), where (p) is a prime number and (n) a positive integer, is of interest because such groups form the building blocks in the classification of finite Abelian groups. The number of non-isomorphic Abelian groups of order (p^n) is closely related to the theory of integer partitions. Therefore, research into the structural characteristics of finite Abelian groups of order (p^n) through invariant factors and elementary divisors is important in order to provide a deeper understanding of the relationship between the internal structure of groups and partition theory.

2. LITERATURE REVIEW

2.1 Basic Concepts of Groups

A group is one of the fundamental structures in abstract algebra, consisting of a non-empty set equipped with a binary operation. Formally, a set (G) with a binary operation $*$ is called a group if it satisfies four axioms: it is closed, associative, has an identity element, and every element has an inverse. Thus, the pair $((G,*))$ is called a group if, for every $(a,b,c \in G)$, the following holds:

1. $(a*b \in G)$ closed);
2. $((a*b)*c = a*(b*c))$ (associative);
3. there exists an identity element $(e \in G)$ such that $(a*e = e*a = a)$;
4. for every $(a \in G)$ there exists $a(a^{-1} \in G)$ such that $(a*a^{-1} = a^{-1}*a = e)$.

If the binary operation on the group also satisfies the property $a*b = b*\mathrm{a}, \forall a,b \in G$ (commutative), then the group is called an Abelian group. Abelian groups have a simpler structure than non-Abelian groups, thus allowing for a complete classification up to isomorphism [1, 2].

2.2 Finite Abelian Groups

A finite Abelian group is an Abelian group with a finite number of elements. The number of elements in a group (G) is called the order of the group and is denoted by $(|G|)$. If

$$|G| = p^n \tag{1}$$

where (p) is a prime number and (n) a positive integer, then the group is called a prime Abelian group or a $(p)-$Abelian group. The study of groups of order (p^n) is very important because these groups form the basic building blocks in the classification of all finite Abelian groups [3].

2.3 Cyclic Groups

A cyclic group is a group that can be generated by a single element. A group (G) is called a cyclic group if there exists an element $(a \in G)$ such that

$$G = \langle a \rangle = a^n \mid n \in Z \tag{2}$$

The element (a) is called a generator of the group. If the cyclic group has finite order (n), then the group is isomorphic to the group of integers modulo (n), namely

$$G \cong Z_n \tag{3}$$

Cyclic groups play an important role in the classification of finite Abelian groups because every finite Abelian group can be expressed as a direct product of several cyclic groups [4].

2.4 The Fundamental Theorem of Finite Abelian Groups

The Fundamental Theorem of Finite Abelian Groups is a key result that describes the structure of all finite Abelian groups. This theorem states that every finite Abelian group can be expressed as a direct product of cyclic groups of certain orders, which are unique up to isomorphism.

Theorem 2.1. (Fundamental Theorem of Finite Abelian Groups)

If (G) is a finite Abelian group, then there exists a positive integer $n_1, n_2, \ldots, n_k$ such that

$$G \cong Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}, \tag{4}$$

where $\mathrm{n_1}|\mathrm{n_2}|\ldots|\mathrm{n_k}$.

This representation is unique up to the order of its factors and is known as the Invariant Factor Decomposition [3]. The theorem can also be stated in another form known as the Elementary Divisor Decomposition, namely as the direct product of cyclic groups of prime powers.

2.5 Invariant Factor

Invariant Factor Decomposition is a representation of a finite abelian group in the form of a direct product of cyclic groups satisfying the sequential divisibility relation. If

$$G \cong Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k},$$

where $n_1|n_2|\ldots|n_k$. then the number$(n_1, n_2, \ldots, n_k)$ is called an invariant factor of the group (G).

This representation has several important characteristics. Firstly, its form is unique up to isomorphism. Secondly, the structure of the group can be observed hierarchically through the divisibility relations between factors. Thirdly, this form facilitates the determination of the group's exponent and various other algebraic properties. Therefore, invariant factor decomposition is often used to provide a global picture of the structure of a finite abelian group [4].

2.6 Elementary Divisor

In addition to using invariant factors, finite Abelian groups can be represented via elementary divisor decomposition. In this form, the group is expressed as the direct product of cyclic groups of prime order, namely

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}} \tag{5}$$

where each (p_i) is a prime number and (a_i) is a positive integer.

This representation provides more detailed information regarding the prime structure of a group. Each factor indicates the direct contribution of a specific prime to the group's structure. Consequently, elementary divisor decomposition is highly useful in the analysis of prime subgroups and the classification of groups of order(p^n)[3].

2.7 Partitions of Integers

Partitions of integers are an important concept in combinatorics and number theory. A partition of a positive integer (n) is defined as a way of expressing (n) as the sum of positive integers, without regard to the order in which they are added. For example, the partitions of the number 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

Thus, there are five partitions of the number 4. The number of partitions of an integer (n) is denoted by (p^n). The partition function plays an important role in various branches of mathematics, including group theory, as it is directly related to the classification of finite Abelian groups of order (p^n) [5].

2.8 The Relationship between Invariant Factors and Elementary Divisors

The relationship between the theory of number partitions and the classification of finite Abelian groups of order(p^n) is a direct consequence of the Fundamental Theorem of Finite Abelian Groups. If

$$|G| = p^n,$$

then every finite Abelian group of order(p^n) can be expressed as

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}}$$

where $a_1 + a_2 + \cdots + a_k = n$.

This equation shows that every partition of the number (n) yields exactly one isomorphism class of finite Abelian groups of order(p^n). Conversely, every finite Abelian group of order(p^n) can be associated with a partition of (n). Therefore, the number of non-isomorphic finite Abelian groups of order(pn) is equal to the number of partitions of (n), namely

$$N(n) = p(n) \tag{6}$$

N(n) = p(n). This result demonstrates a close connection between group theory and the theory of number partitions. Through this connection, the classification of finite Abelian groups of order(p^n) can be carried out systematically using the concept of integer partitions [5] .

3. METHODOLOGY

This research is a qualitative study employing a literature review approach (library research). This approach was chosen because the subject matter consists of concepts, definitions, theorems and mathematical results relating to the structure of finite Abelian groups of order(p^n), particularly in the form of *invariant factor decomposition* and *elementary divisor decomposition*. A literature review was chosen to gain a comprehensive understanding of the classification of finite Abelian groups based on the Fundamental Theorem of Finite Abelian Groups and its relationship with the theory of partitions of integers.

The data sources in this study consist of primary and secondary data. Primary data were obtained from key literature on group theory and abstract algebra, specifically the books **Abstract Algebra** by Dummit and Foote (2004), **An Introduction to the Theory of Groups** by Rotman (2012), **Algebra** by Hungerford (1974), **Contemporary Abstract Algebra** by Gallian (2021), and **Topics in Algebra** by Herstein (1975). Meanwhile, secondary data was obtained from scientific articles, research journals, conference proceedings, and other references relevant to the classification of finite Abelian groups, the theory of number partitions, and the decomposition of finite Abelian groups.

Data collection was carried out through the identification, selection, and review of literature relevant to the research topic. The collected literature was subsequently classified according to subject matter, namely the basic concepts of groups and Abelian groups, cyclic groups, the Fundamental Theorem of Finite Abelian Groups, invariant factor decomposition, elementary divisor decomposition, and the theory of partitions of integers. The selected literature was then analysed to identify conceptual and mathematical connections that support the research objectives.

The data analysis was conducted using a descriptive-analytical approach, examining definitions, theorems and propositions relating to the structure of finite Abelian groups. The initial stage of the analysis involved examining the Fundamental Theorem of Finite Abelian Groups as the basis for the classification of groups. Subsequently, an analysis was carried out on the forms of *invariant factor decomposition* and *elementary divisor decomposition* to identify the characteristics of each representation and their equivalence relations. At this stage, the divisibility properties of *invariant factors* and the prime structure of *elementary divisors* were investigated to demonstrate that both forms represent the same group up to isomorphism.

The proof procedure in this study was carried out using a deductive approach based on established theorems in group theory. The proof began with the application of the Fundamental Theorem of Finite Abelian Groups to show that every finite abelian group can be expressed as a direct product of cyclic groups of prime order. Next, we analyse how the *elementary divisor decomposition* can be transformed into an invariant factor decomposition and vice versa. This process is used to explain the relationship between the two classification methods and to demonstrate the uniqueness of the group's representation up to isomorphism.

To study finite Abelian groups of order p^n, a classification is carried out based on partitions of the integer n. Let a finite Abelian group have order

$$|G| = p^n,$$

where p is a prime number and n is a positive integer. By the Fundamental Theorem of Finite Abelian Groups, the group can be written as

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}}$$

where $a_1 + a_2 + \cdots + a_k = n$.

Each partition of the number n yields a class of isomorphisms of finite Abelian groups of order (p^n). Therefore, the classification process involves identifying all partitions of the number n, then constructing the corresponding group forms and analysing their representations in terms of invariant factors and elementary divisors.

The final stage of the research involves synthesising all the results of the analysis to obtain the general characteristics of finite Abelian groups of order (p^n) . These characteristics include the form of the group's decomposition, the relationship between invariant factors and elementary divisors, and their connection to the theory of partitions of integers. The results of this synthesis are then used to draw conclusions regarding the structure and classification of finite Abelian groups of order (p^n) in a systematic and comprehensive manner.

4. RESULT & DISCUSSION

4.1 Structure of Finite Abelian Groups of Order (p^n)

The main foundation for the classification of finite Abelian groups is the Fundamental Theorem of Finite Abelian Groups. This theorem states that every finite Abelian group can be expressed as a direct product of cyclic groups of prime powers. Formally, if (G) is a finite Abelian group of order

$$|G| = p^n,$$

then there exist positive integers $(a_1, a_2, \cdots, a_k)$ such that

$$a_1 + a_2 + \cdots + a_k = n$$

and

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}}$$

This theorem shows that the structure of a finite Abelian group of order (p^n) is entirely determined by the partitions of the integer (n). In other words, every partition of (n) yields a distinct isomorphism class of finite Abelian groups.

4.2 The Relationship between Partitions of Integers and the Classification of Groups of Order (p^n)

Let (n) be a positive integer. If

$$n = a_1 + a_2 + \cdots + a_k,$$

is a partition of (n), then there exists a finite Abelian group

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}}$$

of order (p^n).

Consequently, the number of non-isomorphic finite Abelian groups of order(pn) is equal to the number of partitions of (n), namely

$$N(n) = p(n)$$

. This result shows that the problem of classifying finite Abelian groups of order (p^n) can be reduced to the problem of determining all partitions of the integer (n).

4.3 Classification of Finite Abelian Groups for Order (p^1), (p^2), (p^3), and (p^4)

Case $(n = 1)$

The partitions of 1 are 1. Thus, there is only one finite Abelian group of order (p), namely

$$Z_p$$

therefore,

$$p(1) = 1$$

Case $(n = 2)$

The partitions of 2 are

$$2, \quad 1+1.$$

The corresponding group is

$$Z_{p^2}$$

and

$$Z_p \oplus Z_p$$

Therefore,

$$p(2) = 2$$

Cases ($n = 3$)

The partitions of 3 are

$$3, \quad 2, \quad 1+1+1.$$

The corresponding groups are

$$Z_p, \quad Z_{p^2} \oplus Z_p, \quad Z_p \oplus Z_p \oplus Z_p$$

Therefore,

$$p(3) = 3$$

Case ($n = 4$)

A partition of 4 is

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1+1, \quad 1+1+1+1$$

The corresponding groups are

$$Z_p, \qquad Z_{p^3} \oplus Z_p, \qquad Z_{p^2} \oplus Z_{p^2}, \qquad Z_{p^2} \oplus Z_p \oplus Z_p, \qquad Z_p \oplus Z_p \oplus Z_p \oplus Z_p$$

Thus, we obtain

$$p(4) = 5$$

.

4.4 Classification Table of Finite Abelian Groups of Order (p^n)

Table 1. Classification Table of Finite Abelian Groups of Order p^n

Value of n	Partitions of the Number	Non-isomorphic Abelian Groups	Number of Groups
1	(1)	(Z_p)	1
2	(2), (1+1)	(Z_{p^2}), $(Z_p \oplus Z_p)$	2
3	(3), (2+1), (1+1+1)	Z_{p^3}, $Z_{p^2} \oplus Z_p$, $Z_p \oplus Z_p \oplus Z_p$	3
4	(4), (3+1), (2+2), (2+1+1), (1+1+1+1)	Five distinct isomorphism classes	5

The table shows that the number of non-isomorphic finite Abelian groups of order(pn) depends on the number of partitions of (n), not on the prime number (p)

4.5 The Relationship between Invariant Factors and Elementary Divisors

The Fundamental Theorem of Finite Abelian Groups can be expressed in two equivalent forms: the invariant factor decomposition and the elementary divisor decomposition. The invariant factor form is written as

$$G \cong Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$$

where

$$\mathrm{n}_1 | \mathrm{n}_2 | \ldots | \mathrm{n}_\mathrm{k}.$$

Meanwhile, the elementary divisor form is written as

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}}$$

The invariant factor representation provides an overview of the group's global structure through divisibility relations, whilst the elementary divisor representation displays the primary structure in greater detail based on its constituent prime factors.

4.6 Example of Conversion from Invariant Factor to Elementary Divisor

Let us consider a group

$$G \cong Z_{12} \oplus Z_{60}$$

the prime factorisation of each order is

$$12 = 2^2 \cdot 3, \qquad 60 = 2^2 \cdot 3 \cdot 5$$

then we obtain

$$Z_{12} \cong Z_4 \oplus Z_3$$

and

$$Z_{60} \cong Z_4 \oplus Z_3 \oplus Z_5$$

therefore,

$$G \cong Z_4 \oplus Z_4 \oplus Z_3 \oplus Z_3 \oplus Z_5$$

The final form is the *elementary divisor decomposition* of the group.

4.7 Discussion

The results of this research show that the structure of finite Abelian groups of order (p^n) follows a systematic classification pattern and is entirely determined by the partitions of the integer n. Based on the Fundamental Theorem of Finite Abelian Groups, every finite Abelian group of order (p^n) can be expressed as the direct product of cyclic groups of prime powers. Consequently, the process of classifying groups no longer depends on the analysis of individual group structures, but can be carried out by identifying all partitions of the number n. This result demonstrates a close connection between group theory and combinatorics, particularly the theory of number partitions.

The classification obtained for the order p, p_2, p_3, and p_4 shows that the number of non-isomorphic finite Abelian groups is, in order, 1, 2, 3 and 5. These numbers correspond to the values of the partition function $p(n)$, namely the number of partitions of the integersn . Thus, the results of this study confirm that the number of isomorphism classes of finite Abelian groups of orderpn is equal to the number of partitions of n. This fact demonstrates that the structure of finite Abelian groups can be understood through a relatively simple combinatorial approach, as opposed to a direct constructive approach.

An analysis of invariant factor decomposition shows that this representation provides a global picture of the group's structure through divisibility relations

$$\mathrm{n}_1 | \mathrm{n}_2 | \ldots | \mathrm{n}_\mathrm{k}$$

This form facilitates the identification of the group's exponent and the presentation of the group's structure in a more concise form. Conversely, elementary divisor decomposition provides more detailed information regarding the prime factors that make up the group. Through this representation, the primary structure of a group can be observed directly, thereby facilitating the analysis of the group's primary subgroups.

The research findings also show that invariant factor decomposition and elementary divisor decomposition are two equivalent representations. Although they take different forms, both describe the same group up to isomorphism. For example, a group

$$G \cong Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$$

can be transformed into elementary divisor form as

$$G \cong Z_{p_1^{a_1}} \oplus Z_{p_2^{a_2}} \oplus \cdots \oplus Z_{p_t^{a_t}}$$

This conversion demonstrates that invariant factor decomposition places greater emphasis on the group's global structure, whilst elementary divisor decomposition reveals its internal structure based on its constituent prime factors.

The findings of this study are consistent with the theory put forward by Dummit and Foote [3], Rotman [4], and Hungerford [6], which states that every finite abelian group has a unique decomposition up to isomorphism. Furthermore, the relationship between the classification of finite Abelian groups and the theory of number partitions obtained in this study is also consistent with classical results in the theory of finite groups. Thus, the results of this study not only reinforce the validity of the

Fundamental Theorem of Finite Abelian Groups, but also demonstrate how concepts in number theory and combinatorics can be used to systematically explain the structure of finite Abelian groups.

Theoretically, the results of this research provide a deeper understanding of the characteristics of finite Abelian groups of order (p^n) and can serve as a basis for further study, such as the theory of finite modules over Principal Ideal Domains (PIDs), group representation theory, and the classification of other algebraic structures with similar decomposition characteristics. Consequently, the relationship between invariant factor decomposition, elementary divisor decomposition, and the theory of number partitions constitutes a key aspect in the development of modern group theory and algebra.

5. CONCLUSION

Every finite Abelian group of order(pn) can be classified based on the partitions of the integer n. The number of non-isomorphic finite Abelian groups of order (p^n) is equal to the number of partitions of n, namely $p(n)$. Furthermore, invariant factor decomposition and elementary divisor decomposition are two equivalent representations for describing the structure of finite Abelian groups, yet they provide different perspectives on the internal characteristics of the group.

The findings of this research are consistent with the theory put forward by Dummit and Foote (2004), Rotman (2012), and Hungerford (1974), which states that every finite Abelian group has a unique decomposition up to isomorphism. Furthermore, the relationship between the classification of finite Abelian groups and the theory of number partitions obtained in this study is also consistent with classical results in the theory of finite groups. Thus, the results of this study not only reinforce the validity of the Fundamental Theorem of Finite Abelian Groups, but also demonstrate how concepts from number theory and combinatorics can be used to systematically explain the structure of finite Abelian groups.Theoretically, the results of this research provide a deeper understanding of the characteristics of finite Abelian groups of order(pn) and can serve as a basis for further study, such as the theory of finite modules over Principal Ideal Domains (PIDs), group representation theory, or the classification of other algebraic structures with similar decomposition characteristics. Therefore, the relationship between invariant factor decomposition, elementary divisor decomposition, and the theory of number partitions is a key aspect in the development of modern group theory and algebra.

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